

MINIMAX RATES FOR ESTIMATING THE DIMENSION OF A MANIFOLD

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ABSTRACT. Many algorithms in machine learning and computational geometry require, as input, the intrinsic dimension of the manifold that supports the probability distribution of the data. This parameter is rarely known and therefore has to be estimated. We characterize the statistical difficulty of this problem by deriving upper and lower bounds on the minimax rate for estimating the dimension. First, we consider the problem of testing the hypothesis that the support of the data-generating probability distribution is a well-behaved manifold of intrinsic dimension d_1 versus the alternative that it is of dimension d_2 , with $d_1 < d_2$. With an i.i.d. sample of size n , we provide an upper bound on the probability of choosing the wrong dimension of $O(n^{-(d_2/d_1-1-\epsilon)n})$, where ϵ is an arbitrarily small positive number. The proof is based on bounding the length of the traveling salesman path through the data points. We also demonstrate a lower bound of $\Omega(n^{-(2d_2-2d_1+\epsilon)n})$, by applying Le Cam's lemma with a specific set of d_1 -dimensional probability distributions. We then extend these results to get minimax rates for estimating the dimension of well-behaved manifolds. We obtain an upper bound of order $O(n^{-(\frac{1}{m-1}-\epsilon)n})$ and a lower bound of order $\Omega(n^{-(2+\epsilon)n})$, where m is the embedding dimension.

1 Introduction

Suppose that X_1, \dots, X_n is an i.i.d. sample from a distribution P whose support is an unknown, well behaved, manifold M of dimension d in \mathbb{R}^m , where $1 \leq d \leq m$. Manifold learning refers broadly to a suite of techniques from statistics and machine learning aimed at estimating M or some of its features based on the data.

Manifold learning procedures are widely used in high dimensional data analysis, mainly to alleviate the curse of dimensionality. Such algorithms map the data to a new, lower dimensional coordinate system [Bellman, 1961, Lee and Verleysen, 2007a, Hastie et al.,

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2009], with little loss in accuracy. Manifold learning can greatly reduce the dimensionality of the data.

Most manifold learning techniques require, as input, the intrinsic dimension of the manifold. However, this quantity is almost never known in advance and therefore has to be estimated from the data.

Various intrinsic dimension estimators have been proposed and analyzed; [see, e.g., Lee and Verleysen, 2007b, Koltchinskii, 2000, Kégl, 2003, Levina et al., 2004, Hein and Audibert, 2005, Raginsky and Lazebnik, 2005, Little et al., 2009, 2011, Sricharan et al., 2010, Rozza et al., 2012, Camastra and Staiano, 2016]. However, characterizing the intrinsic statistical hardness of estimating the dimension remains an open problem.

The traditional way of measuring the difficulty of a statistical problem is to bound its *minimax risk*, which in the present setting is loosely described as the worst possible statistical performance of an optimal dimension estimator. Formally, given a class of probability distribution \mathcal{P} , the minimax risk $R_n = R_n(\mathcal{P})$ is defined as

$$R_n = \inf_{\hat{d}} \sup_{P \in \mathcal{P}} \mathbb{E}_P \left[1(\hat{d} \neq d(P)) \right]. \quad (1.1)$$

In Equation (1.1), $d(P)$ is the dimension of the support of P , \mathbb{E}_P denotes the expectation with respect to the distribution P , $1(\cdot)$ is the indicator function, and the infimum is over all estimators (measurable functions of the data) $\hat{d} = \hat{d}(X_1, \dots, X_n)$ of the dimension $d(P)$. The risk $\mathbb{E}_P[1(\hat{d} \neq d(P))]$ of a dimension estimator \hat{d} is the probability that \hat{d} differs from the true dimension $d(P)$ of the support of the data generating distribution P . The minimax risk $R_n(\mathcal{P})$, which is a function of both the sample size n and the class \mathcal{P} , quantifies the intrinsic hardness of the dimension estimation problem, in the sense that *any dimension estimator* cannot have a risk smaller than R_n uniformly over every $P \in \mathcal{P}$.

The purpose of this paper is to obtain upper and lower bounds on the minimax risk R_n in (1.1). We impose several regularity conditions on the set of manifolds supporting the distribution in the class \mathcal{P} , in order to make the problem analytically tractable and also to avoid pathological cases, such as space-filling manifolds. We first assume that the manifold supporting the data generating distribution P has two possible dimensions, d_1 and d_2 . This assumption is then relaxed to any dimension $d(P)$ between 1 and the embedding dimension m . Our main result is the following theorem. See Section 2 for the definition of the class \mathcal{P} of probability distributions supported on well-behaved manifolds in \mathbb{R}^m .

Theorem 1. *The minimax risk R_n in (1.1) satisfies, $a_n \leq R_n \leq b_n$, where*

$$a_n = (C_{K_I}^{(15)})^n \tau_\ell^n \min\{\tau_\ell^{-3} n^{-2}, 1\}^n, \quad (1.2)$$

$$b_n = (C_{K_I, K_p, K_v, m}^{(14)})^n (1 + \tau_g^{-(m^2-m)n}) n^{-\frac{n}{m-1}}, \quad (1.3)$$

and the constants τ_ℓ , τ_g , $C_{K_I}^{(15)}$ and $C_{K_I, K_p, K_v, m}^{(14)}$ depend on \mathcal{P} and are defined in Section 5.

We now make a few remarks about the previous theorem.

- Since the dimension $d(P)$ is a discrete quantity, the minimax rate R_n in (1.1) is superexponential in sample size. This result seems at odds with the exponential rate obtained by [Koltchinskii, 2000, Proposition 2.1]. These different rates are due to different model assumptions. In [Koltchinskii, 2000] the data generating distribution is the convolution of a probability distribution supported on a manifold with a noise distribution supported on a set of full dimension m . In contrast, here we assume that the data are generated from a probability distribution supported on a manifold. Under our noiseless model, distributions supported on manifolds with different dimension are more easily distinguishable, hence the minimax rate R_n converges to 0 faster than under the model with noise assumed by [Koltchinskii, 2000].
- The key quantities that appear in the lower bound (1.2) and the upper bound (1.3) are the global reach τ_g and the local reach τ_ℓ of the manifold, which are defined in Section 2. These reach parameters can be roughly thought as the inverse of the usual notion of curvature [see, e.g. Federer, 1959], and they affect the performance of any dimension estimator: a manifold with low reach may appear more space-filling than a manifold of the same dimension but with higher reach, thus making the task of resolving the dimension harder. Indeed, our analysis shows formally that the minimax risk R_n in (1.1) decreases in the values of the reaches. Given their crucial role, we have attempted to make the dependence of the minimax risk R_n on both τ_g and τ_ℓ as explicit as possible.
- There is a gap between the lower bound (1.2) and the upper bound (1.3). Nonetheless, as far as we are aware, these are the most precise bounds on R_n that are available.

This paper is organized as follows. In Section 2, we formulate and discuss regularity conditions on distributions and their supporting manifolds. In Section 3, we provide an upper bound on the minimax rate by considering the traveling salesman path through the points. In Section 4, we derive a lower bound on the minimax rate by applying Le Cam’s lemma with a specific set of d_1 -dimensional and d_2 -dimensional probability distributions. In Section 5, we extend our upper bound and lower bound for the case where the intrinsic dimension varies from 1 to m .

2 Definitions and Regularity Conditions

In this section, we define the set \mathcal{P} of probability distributions that we consider in bounding the minimax risk R_n in (1.1). Such distributions are supported on manifolds whose dimension d is between 1 and m , where m is the dimension of the embedding space. In

particular, we require that the supporting manifolds have a uniform lower bound on their reach parameters τ_g and τ_ℓ . The resulting class of distributions is denoted by

$$\mathcal{P} = \bigcup_{d=1}^m \mathcal{P}_{\tau_g, \tau_\ell, K_I, K_v, K_p}^d. \quad (2.1)$$

In the rest of this section, we will make the definition $\mathcal{P}_{\tau_g, \tau_\ell, K_I, K_v, K_p}^d$ precise. Readers who are not interested in the details may skip the rest of the section.

2.1 Notation and Basic Definitions

For the reader's convenience, we provide a list of the notation used throughout the paper in Table 1.

We now briefly review some notations from differential geometry. For a more detailed treatment, we refer the reader to standard textbooks on this topic [see, e.g., Lee, 2000, 2003, Petersen, 2006, do Carmo, 1992]. A topological manifold of dimension d is a topological space M and a family of homeomorphisms $\varphi_\alpha : U_\alpha \subset \mathbb{R}^d \rightarrow V_\alpha \subset M$ from an open subset of \mathbb{R}^d to an open subset of M such that $\bigcup_\alpha \varphi_\alpha(U_\alpha) = M$. A topological space M is considered to be a d -dimensional manifold if there exists a family of homeomorphisms $\varphi_\alpha : U_\alpha \subset \mathbb{R}^d \rightarrow V_\alpha \subset M$ such that $(M, \{\varphi_\alpha\}_\alpha)$ is a manifold. If M is a d -dimensional manifold, such d is unique and is called the dimension of a manifold. If, for any pair α, β , with $\varphi_\alpha(U_\alpha) \cap \varphi_\beta(U_\beta) \neq \emptyset$, $\varphi_\beta^{-1} \circ \varphi_\alpha : U_\alpha \cap U_\beta \rightarrow U_\alpha \cap U_\beta$ is C^k , then M is a C^k -manifold.

We assume that the topological manifold M is embedded in \mathbb{R}^m , i.e. $M \subset \mathbb{R}^m$, and the metric is inherited from the metric of \mathbb{R}^m . For a topological manifold $M \subset \mathbb{R}^m$ and for any $p, q \in M$, a path joining p to q is a map $\gamma : [a, b] \rightarrow M$ for some $a, b \in \mathbb{R}$ such that $\gamma(a) = p$, $\gamma(b) = q$. The length of the curve γ is defined as $Length(\gamma) = \int_a^b \|\gamma'(t)\|_2 dt$. A topological manifold M is equipped with the distance $dist_M : M \times M \rightarrow \mathbb{R}$ as $dist_M(p, q) = \inf_{\gamma: \text{path joining } p \text{ and } q} Length(\gamma)$. A path $\gamma : [a, b] \rightarrow M$ is a geodesic if for all $t, t' \in [a, b]$, $dist_M(\gamma(t), \gamma(t')) = |t - t'|$.

Let $T_p M$ denote the tangent space to M at p . Given $p \in M$, there exist a set $0 \in \mathcal{E} \subset T_p(M)$ and a mapping $\exp_p : \mathcal{E} \subset T_p M \rightarrow M$ such that $t \rightarrow \exp_p(tv)$, $t \in (-1, 1)$, is the unique geodesic of M which, at $t = 0$, passes through p with velocity v , for all $v \in \mathcal{E}$. The map $\exp_p : \mathcal{E} \subset T_p M \rightarrow M$ is called the exponential map on p .

One of the key conditions that we impose in Section 2.3 is about the reach.

Definition 1. For a compact d -dimensional topological manifold $M \subset \mathbb{R}^m$ (with boundary), the *reach* of M , $\tau(M)$, is defined as the largest value of r such that for all $x \in \mathbb{R}^m$

Notation	Definition
$1(\cdot)$	indicator function.
d, d_1, d_2	dimension of a manifold.
\hat{d}	dimension estimator.
$dist_A(\cdot, \cdot)$	distance function on the set A .
$dist_{A, \ \cdot\ }(\cdot, \cdot)$	distance function on the set A induced by the norm $\ \cdot\ $.
$\exp_p(\cdot)$	exponential map on point $p \in M$.
$\ell(\cdot, \cdot)$	loss function.
n	size of the sample.
m	dimension of the embedding space.
p, q	points on the manifold M .
$vol_A(\cdot)$	volume function of A .
$B_A(x, r)$	open ball with center x and radius r , $\{y \in A : dist_A(y, x) < r\}$.
C_{a_1, \dots, a_k}	constants that depends only on a_1, \dots, a_k .
I	cube $[-K_I, K_I]^m$.
K_I, K_v, K_p	fixed constants for regular conditions; see Definition 2.
M	manifold.
P	data generating probability distribution.
R_n	minimax risk $\inf_{\hat{d}} \sup_{P \in \mathcal{P}} \mathbb{E}_P \left[1 \left(\hat{d} \neq d(P) \right) \right]$; see (1.1), (2.5), and (2.6).
S_n	permutation group on $\{1, \dots, n\}$.
T	subset of $I^n \subset (\mathbb{R}^d)^n$, used in Section 4.
$T_p M$	tangent space of a manifold M at p .
X_1, \dots, X_n	sample points.
\mathcal{M}	set of manifolds; see Definition 2.
\mathcal{P}	set of distributions; see Definition 2.
γ	path on a manifold M .
$\pi_A(\cdot)$	projection function onto a closed set A .
σ	permutation.
$\tau(M)$	reach of a manifold M ; see Definition 1 and Lemma 2.
τ_g	lower bound for global reach; see Definition 2.
τ_ℓ	lower bound for local reach; see Definition 2.
ω_d	volume of the unit ball in \mathbb{R}^d , $\frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}$.
$\Pi_{n_1:n_2}$	coordinate projection map: $\Pi_{n_1:n_2}(x_1, \dots, x_d) = (x_{n_1}, \dots, x_{n_2})$.

Table 1: Table of notations and definitions.

with $dist_{\mathbb{R}^m}(x, M) < r$ has the unique projection $\pi_M(x)$ to M , i.e.

$$\tau(M) := \sup \left\{ r : \forall x \in \mathbb{R}^m \text{ with } dist_{\mathbb{R}^m}(x, M) < r, \right. \\ \left. \exists! \pi_M(x) \in M \text{ s.t. } \|x - \pi_M(x)\|_2 = \inf_{y \in M} \|x - y\|_2 \right\}. \quad (2.2)$$

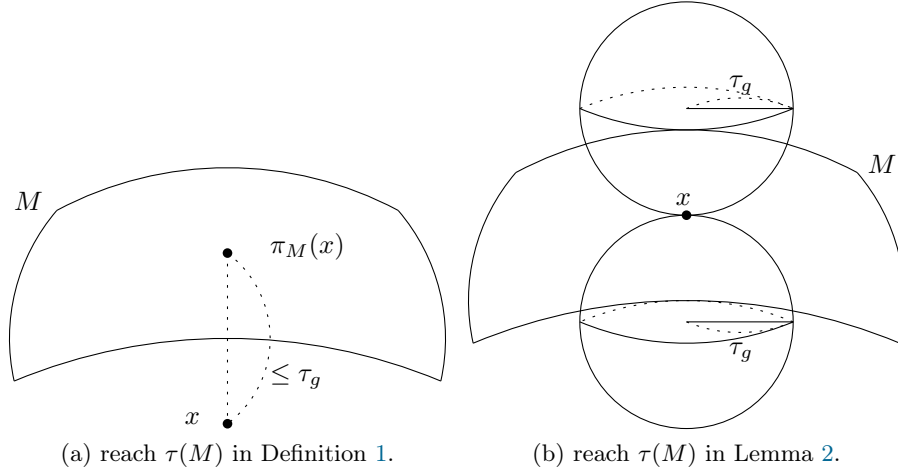


Figure 2.1: For a manifold M , there are several equivalent definitions for reach $\tau(M)$ in Definition 1. (a) The reach $\tau(M)$ is the supremum value of r such that for all $x \in \mathbb{R}^m$ with $\text{dist}_{\mathbb{R}^m}(x, M) < r$ has unique projection $\pi_M(x)$ to M , as in (2.2). (b) The reach $\tau(M)$ is the maximum radius of a ball that you can roll over the manifold M , as in (2.3).

See [Federer, 1959] for further details. The reach $\tau(M)$ can be also considered as one kind of curvature, and can be understood as an inverse of other usual curvatures. See Figure 2.1(a) for the illustration of Definition 1. There are several equivalent ways to define the reach $\tau(M)$ in (2.2) for the manifold M . The reach $\tau(M)$ is the maximum radius of a ball that can be rolled freely over the manifold M , as in Lemma 2. See Figure 2.1(b) for the illustration of Lemma 2.

Lemma 2. For a manifold $M \subset \mathbb{R}^m$,

$$\tau(M) = \sup\{r : \forall x \in M, \forall y \in M \text{ with } y - x \perp T_x M \text{ and } \|y - x\|_2 = r, \\ B_{\mathbb{R}^m}(y, r) \cap M = \emptyset\}. \quad (2.3)$$

Proof. [See Federer, 1959, Theorem 4.18]. □

2.2 Minimax Theory

The minimax rate is the risk of an estimator that performs best in the worst case, as a function of the sample size [see, e.g. Tsybakov, 2008]. Let \mathcal{P} be a collection of probability distributions over the same sample space \mathcal{X} and let $\theta : \mathcal{P} \rightarrow \Theta$ be a function over \mathcal{P} taking

values in some space Θ , the parameter space. We can think of $\theta(P)$ as the feature of interest of the probability distribution P , such as its mean, or, as in our case, the dimension of its support. For the fixed sample size n , suppose $X = (X_1, \dots, X_n)$ is an i.i.d. (independent and identically distributed) sample drawn from a fixed probability distribution $P \in \mathcal{P}$. Thus X takes values in the n -fold product space $\mathcal{X}^n = \mathcal{X} \times \dots \times \mathcal{X}$ and is distributed as $P^{(n)}$, the n -fold product measure. An estimator $\hat{\theta}_n : \mathbb{R}^n \rightarrow \Theta$ is any measurable function that maps the observation X into the parameter space Θ . Let $\ell : \Theta \times \Theta \rightarrow \mathbb{R}$ be a loss function, a non-negative, non-decreasing bounded function that measures how different two parameters are. Then for a fixed estimator $\hat{\theta}_n$ and a fixed distribution P , the risk of $\hat{\theta}_n$ is defined as

$$\mathbb{E}_{P^{(n)}} \left[\ell \left(\hat{\theta}_n(X), \theta(P) \right) \right].$$

Then for a fixed estimator $\hat{\theta}_n$, its maximum risk is the supremum of its risk over every distribution $P \in \mathcal{P}$, that is,

$$\sup_{P \in \mathcal{P}} \mathbb{E}_{P^{(n)}} \left[\ell \left(\hat{\theta}_n(X), \theta(P) \right) \right]. \quad (2.4)$$

The minimax risk associated to \mathcal{P} , θ , ℓ and n is the maximal risk of any estimator that performs the best under the worst possible choice of P . Formally, the *minimax risk* is

$$R_n = \inf_{\hat{\theta}_n} \sup_{P \in \mathcal{P}} \mathbb{E}_{P^{(n)}} \left[\ell \left(\hat{\theta}_n(X), \theta(P) \right) \right]. \quad (2.5)$$

The minimax risk R_n in (2.5) is often viewed as a function of the sample size n , in which case any positive sequence ψ_n such that $\lim_{n \rightarrow \infty} R_n / \psi_n$ remains bounded away from 0 and ∞ is called a *minimax rate*. Notice that minimax rates are unique up to constants and lower order terms.

To define a meaningful minimax risk, it is essential to have some constraint on the set of distributions \mathcal{P} in (2.4) and (2.5). If \mathcal{P} is too large, then the minimax rate R_n in (2.5) will not converge to 0 as n goes to ∞ : this means that the problem is statistically ill-posed. If \mathcal{P} is too small, the minimax estimator depends too much on the specific distributions in \mathcal{P} and is not a useful measure of a statistical difficulty.

Determining the value of the minimax risk R_n in (2.5) for a given problem requires two separate calculations: an upper bound on R_n and a lower bound. In order to derive an upper bound, one analyzes the asymptotic risk of a specific estimator $\hat{\theta}_n$. Lower bounds are instead usually computed by measuring the difficulty of a multiple hypothesis testing problem that entails identifying finitely many distributions in \mathcal{P} that are maximally difficult to discriminate [see, e.g. [Tsybakov, 2008](#), Section 2.2].

For the dimension estimation problem, we obtain an upper bound on R_n by analyzing the performance of an estimator based on the length of the traveling salesman problem, as described in Section 3. On the other hand, the calculation of the lower bound presents

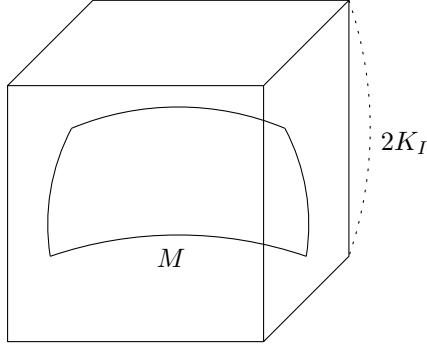


Figure 2.2: A manifold M is assumed to be contained inside the cube $I = [-K_I, K_I]^m$, for some $K_I > 0$. See Definition 2.

non-trivial technical difficulties, because probability distributions supported on manifolds of different dimensions are singular with respect to each other, and therefore trivially discriminable. In order to overcome such an issue, we resort to constructing mixtures of mutually singular distributions. We detail this construction in Section 4.

There is a gap between the lower and upper bounds we derive on the minimax risk, as it is often the case in such calculations. Nonetheless the derivation of the bounds is of use in understanding the difficulty of the dimension estimation problem.

2.3 Regularity conditions on the Distributions and their Supporting Manifolds

In our analysis we require various regularity conditions on the class \mathcal{P} of probability distributions appearing in the minimax risk (1.1). Most of these conditions are of a geometric nature and concern the properties of the manifolds supporting the probability distributions in \mathcal{P} . Altogether, our assumptions rule out manifolds that are so complicated to make the dimension estimation problem unsolvable and, therefore, guarantee that the minimax risk R_n in (2.5) converges to 0 as n goes to ∞ . Such regularity assumptions are quite mild, and in fact allow for virtually all types of manifolds usually encountered in manifold learning problems.

Our first assumption is that the probability distributions in \mathcal{P} are supported over manifold contained inside a compact set, which, without loss of generality, we take to be the cube $I := [-K_I, K_I]^m$, for some $K_I > 0$. See Figure 2.2.

Second, to exclude manifolds that are arbitrarily complicated in the sense of having unbounded curvatures or of being nearly self intersecting, we assume that the reach is uniformly bounded from below. More precisely, we will constrain both the global reach and the local reach as follows. Fix $\tau_g, \tau_\ell \in (0, \infty]$ with $\tau_g \leq \tau_\ell$. The global reach condition for a

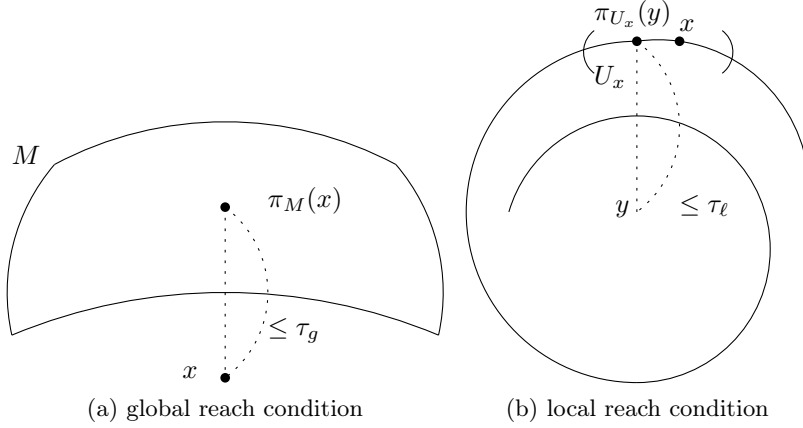


Figure 2.3: A manifold M with (a) *global reach* at least τ_g , or (b) *local reach* at least τ_ℓ . See Definition 2.

manifold M is that the usual reach $\tau(M)$ in (2.2) is lower bounded by τ_g as in Figure 2.3(a), and the local reach condition is that M can be covered by small patches whose reaches are lower bounded by τ_ℓ , as in Figure 2.3(b). (See Definition 2 below for more details.)

Third, we assume that the data are generated from a distribution P supported on a manifold M having a density with respect to the (restriction of the) Hausdorff measure on M bounded from above by some positive constant K_p .

For manifolds without boundary, the above conditions suffice for our analysis. However, to deal with manifolds with boundary, we need further assumptions, namely local geodesic completeness and essential dimension. A manifold M is said to be complete if any geodesic can be extended arbitrarily farther, i.e. for any geodesic path $\gamma : [a, b] \rightarrow M$, there exists a geodesic $\tilde{\gamma} : \mathbb{R} \rightarrow M$ that satisfies $\tilde{\gamma}|_{[a, b]} = \gamma$. [see, e.g., Lee, 2000, 2003, Petersen, 2006, do Carmo, 1992]. Accordingly, we define a manifold M to be locally (geodesically) complete, if any two points inside a geodesic ball of small enough radius in the interior of M can be joined by a geodesic whose image also lies on the interior of M .

Fifth, we assume the manifold M is of essential dimension d , in volume sense. If we fix any point p in the d -dimensional manifold M , then the volume of a ball of radius r grows in order of r^d when r is small. By extending this, fix $K_v \in (0, 2^{-m}]$, and we say that the manifold M is of essential volume dimension d , if the volume of a geodesic ball of radius r around any point in M is lower bounded by $K_v r^d \omega_d$, for some positive constant K_v and all r small enough.

We are now ready to formally define the class \mathcal{P} of probability distributions that we will consider in our analysis of the minimax problem (1.1).

Definition 2. Fix $\tau_g, \tau_\ell \in (0, \infty]$, $K_I \in [1, \infty)$, $K_v \in (0, 2^{-m}]$, with $\tau_g \leq \tau_\ell$. Let $\mathcal{M}_{\tau_g, \tau_\ell, K_I, K_v}^d$ be the set of compact d -dimensional manifolds M such that:

- (1) $M \subset I := [-K_I, K_I]^m \subset \mathbb{R}^m$;
- (2) M is of *global reach* at least τ_g , i.e. $\tau(M) \geq \tau_g$, and M is of *local reach* at least τ_ℓ , i.e. for all $p \in M$, there exists a neighborhood U_p in M such that $\tau(U_p) \geq \tau_\ell$;
- (3) M is *locally (geodesically) complete* (with respect to τ_g): for all $p \in \text{int}(M)$ and for all $q_1, q_2 \in B_M(p, 2\sqrt{3}\tau_g)$, there exists a geodesic γ joining q_1 and q_2 whose image lies on $\text{int}M$;
- (4) M is of *essential volume dimension* d (with respect to K_v and τ_g): if for all $p \in M$ and for all $r \leq \sqrt{3}\tau_g$, $\text{vol}_M(B_M(p, r)) \geq K_v r^d \omega_d$.

Let $\mathcal{P} = \mathcal{P}_{\tau_g, \tau_\ell, K_I, K_v, K_p}^d$ be the set of Borel probability distributions P such that:

- (5) P is supported on a d -dimensional manifold $M \in \mathcal{M}_{\tau_g, \tau_\ell, K_I, K_v}^d$;
- (6) P is absolutely continuous with respect to the restriction vol_M of the d -dimensional Hausdorff measure on the supporting manifold M and such that $\sup_{x \in M} \frac{dP}{d\text{vol}_M}(x) \leq K_p$.

For every $P \in \mathcal{P}_{\tau_g, \tau_\ell, K_I, K_v, K_p}^d$, denote the dimension of its distribution as $d(P)$.

Remark 1. For manifolds without boundary, the local completeness condition and the essential volume dimension condition in Definition 2 always hold. The Hopf Rinow Theorem [see, e.g. Petersen, 2006, Theorem 16] implies that any compact closed manifold without boundary is geodesic complete, which implies it is locally complete in the sense of (3) in Definition 2. Also, [Niyogi et al., 2008, Lemma 5.3] implies that, for a d -dimensional manifold M and all $0 < r \leq 2\tau(M)$,

$$\text{vol}_M(B_M(p, r)) \geq r^d \left(1 - \left(\frac{r}{2\tau(M)} \right)^2 \right)^{\frac{d}{2}} \omega_d,$$

for all $p \in M$. Hence, when, for fixed $\tau_g > 0$, a d -dimensional manifold M (without boundary) satisfies $\tau(M) \geq \tau_g$, then for any $0 < r \leq \sqrt{3}\tau_g$, $\text{vol}_M(B_M(p, r)) \geq 2^{-d} r^d \omega_d$, so the essential volume dimension condition is satisfied.

The regularity conditions in Definition 2 imply further constraints on both the distributions in \mathcal{P} and their supporting manifolds, in Lemma 3, 4, and 5. Such properties are exploited in Section 3 and 4. The proofs for Lemma 3 and Lemma 5 are in Appendix A.

Lemma 3. Fix $\tau_g, \tau_\ell \in (0, \infty]$, $K_I \in [1, \infty)$, $K_v \in (0, 2^{-m}]$, with $\tau_g \leq \tau_\ell$. For $M \in \mathcal{M}_{\tau_g, \tau_\ell, K_I, K_v}^d$ and $r \in (0, \tau_g)$, let $M_r := \{x \in \mathbb{R}^m : \text{dist}_{\mathbb{R}^m}(x, M) < r\}$ be a r -neighborhood of M in \mathbb{R}^m . Then, the volume of M is upper bounded as:

$$\begin{aligned} \text{vol}_M(M) &\leq C_{d,m}^{(3,1)} r^{d-m} \text{vol}_{\mathbb{R}^m}(M_r) \\ &\leq C_{K_I, d, m}^{(3,2)} \left(1 + \tau_g^{d-m}\right), \end{aligned}$$

where $C_{d,m}^{(3,1)}$ is a constant depending only on d and m , and $C_{K_I, d, m}^{(3,2)}$ is a constant depending only on K_I , d and m .

Lemma 4. Fix $\tau_g, \tau_\ell \in (0, \infty]$, $K_I \in [1, \infty)$, $K_v \in (0, 2^{-m}]$, with $\tau_g \leq \tau_\ell$. Let $M \in \mathcal{M}_{\tau_g, \tau_\ell, K_I, K_v}^d$ and $r \in (0, 4\tau_g]$. Then M can be covered by N radius r balls $B_M(p_1, r), \dots, B_M(p_N, r)$, with

$$N = \left\lceil \frac{2^d \text{vol}(M)}{K_v r^d \omega_d} \right\rceil.$$

Proof. [See [Ma and Fu, 2011](#), 4.3.1. Lemma 3]. □

Lemma 5. Fix $\tau_g, \tau_\ell \in (0, \infty]$, $K_I \in [1, \infty)$, $K_v \in (0, 2^{-m}]$, with $\tau_g \leq \tau_\ell$. Let $M \in \mathcal{M}_{\tau_g, \tau_\ell, K_I, K_v}^d$ and let $\exp_{p_k} : \mathcal{E}_k \subset \mathbb{R}^m \rightarrow \mathcal{M}$ be an exponential map, where \mathcal{E}_k is the domain of the exponential map \exp_{p_k} and $T_{p_k}M$ is identified with \mathbb{R}^m . For all $v, w \in \mathcal{E}_k$, let $R_k := \max\{\|v\|, \|w\|\}$. Then

$$\|\exp_{p_k}(v) - \exp_{p_k}(w)\|_{\mathbb{R}^m} \leq \frac{\sinh(\kappa_\ell R_k)}{\kappa_\ell R_k} \|v - w\|_{\mathbb{R}^d}.$$

Under these regularity conditions, the minimax risk R_n is defined as

$$R_n = \inf_{\hat{d}_n} \sup_{P \in \mathcal{P}} \mathbb{E}_{P^{(n)}} \left[1 \left(\hat{d}_n(X) \neq d(P) \right) \right], \quad (2.6)$$

where in Section 3 and 4 we fix $d_1, d_2 \in \mathbb{N}$ with $1 \leq d_1 < d_2 \leq m$ and define

$$\mathcal{P} = \mathcal{P}_{\tau_g, \tau_\ell, K_I, K_v, K_p}^{d_1} \bigcup \mathcal{P}_{\tau_g, \tau_\ell, K_I, K_v, K_p}^{d_2}, \quad (2.7)$$

and in Section 5 we set instead

$$\mathcal{P} = \bigcup_{d=1}^m \mathcal{P}_{\tau_g, \tau_\ell, K_I, K_v, K_p}^d. \quad (2.8)$$

In (2.6), \hat{d}_n is any dimension estimator based on data $X = (X_1, \dots, X_n)$, and the loss function $\ell(\cdot, \cdot)$ is 0-1 loss, so for all $x, y \in \mathbb{R}$, $\ell(x, y) = 1(x \neq y)$.

3 Upper Bound for Choosing Between Two Dimensions

In this section we provide an upper bound on the minimax rate R_n in (2.6) when $d(P)$ can only take two known values. Fix $d_1, d_2 \in \mathbb{N}$ with $1 \leq d_1 < d_2 \leq m$, and assume that the data are generated from a distribution $P \in \mathcal{P}$ such that either $d(P) = d_1$ or $d(P) = d_2$ as in (2.7). In this case, the minimax risk quantifies the statistical hardness of the hypothesis testing problem of deciding whether the data originate from a d_1 or d_2 -dimensional distribution. In Section 5 we will relax this assumption and allow for the intrinsic dimension $d(P)$ to be any integer between 1 and m as in (2.8),

Our strategy to derive an upper bound on R_n is to choose a particular estimator \hat{d}_n and then derive a uniform upper bound on its risk over the class \mathcal{P} in (2.7), i.e. an upper bound for the quantity

$$\sup_{P \in \mathcal{P}} \mathbb{E}_{P^{(n)}} \left[1 \left(\hat{d}_n(X) \neq d(P) \right) \right], \quad (3.1)$$

where $P^{(n)}$ denotes the n -fold product of P . This will in turn yield an upper bound on the minimax risk R_n , since

$$R_n = \inf_{\hat{d}_n} \sup_{P \in \mathcal{P}} \mathbb{E}_{P^{(n)}} \left[1 \left(\hat{d}_n(X) \neq d(P) \right) \right] \leq \sup_{P \in \mathcal{P}} \mathbb{E}_{P^{(n)}} \left[1 \left(\hat{d}_n(X) \neq d(P) \right) \right]. \quad (3.2)$$

Naturally, choosing an appropriate estimator is critical to get a sharp bound. Our estimator \hat{d}_n is based on the d_1 -squared length of the TSP (Traveling Salesman Path) generated by the data. The d_1 -squared length of the TSP generated by the data is the minimal d_1 -squared length of all possible paths passing through each sample point X_i once, which is

$$\min_{\sigma \in S_n} \left\{ \sum_{i=1}^{n-1} \|X_{\sigma(i+1)} - X_{\sigma(i)}\|_{\mathbb{R}^m}^{d_1} \right\}. \quad (3.3)$$

Then, $\hat{d}_n = d_1$ if and only if the d_1 -squared length of the TSP is below a certain threshold; that is

$$\hat{d}_n(X) = d_1 \iff \min_{\sigma \in S_n} \left\{ \sum_{i=1}^{n-1} \|X_{\sigma(i+1)} - X_{\sigma(i)}\|_{\mathbb{R}^m}^{d_1} \right\} \leq C_{K_I, K_v, d_1, m}^{(7)} \left(1 + \tau_g^{d_1-m} \right), \quad (3.4)$$

where $C_{K_I, K_v, d_1, m}^{(7)}$ is a constant to be defined later.

We begin our analysis of the estimator \hat{d}_n with Lemma 6, which shows that \hat{d}_n makes an error with probability of order $O\left(n^{-\left(\frac{d_2}{d_1}-1\right)n}\right)$ if the correct dimension is d_2 . Specifically, we demonstrate that, for any positive value L , the d_1 -squared length of a piecewise linear path from X_1 to X_n , $\sum_{i=1}^{n-1} \|X_{i+1} - X_i\|_{\mathbb{R}^m}^{d_1}$, is upper bounded by L with a

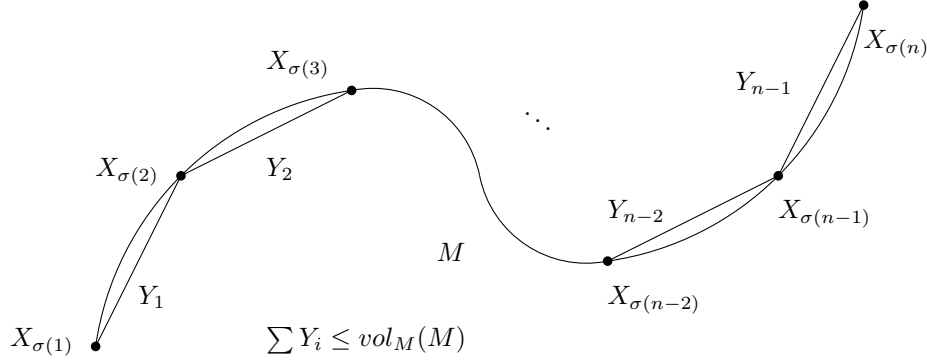


Figure 3.1: When the manifold is a curve, the length of the TSP path $\min_{\sigma \in S_n} \left\{ \sum_{i=1}^{n-1} \|X_{\sigma(i+1)} - X_{\sigma(i)}\|_{\mathbb{R}^m} \right\}$ in (3.3) is upper bounded by the length of the curve $\text{vol}_M(M)$.

very small probability of order $O\left(n^{-\left(\frac{d_2}{d_1}-1\right)n}\right)$, as in (3.5). Hence the d_1 -squared length of the path is not likely to be bounded by any such threshold L .

Lemma 6. Fix $\tau_g, \tau_\ell \in (0, \infty]$, $K_I \in [1, \infty)$, $K_v \in (0, 2^{-m}]$, $K_p \in [(2K_I)^m, \infty)$, $d_1, d_2 \in \mathbb{N}$, with $\tau_g \leq \tau_\ell$ and $1 \leq d_1 < d_2 \leq m$. Let $X_1, \dots, X_n \sim P \in \mathcal{P}_{\tau_g, \tau_\ell, K_I, K_v, K_p}^{d_2}$. Then for all $L > 0$,

$$P^{(n)} \left[\sum_{i=1}^{n-1} \|X_{i+1} - X_i\|^{d_1} \leq L \right] \leq \frac{\left(C_{K_I, K_p, d_1, d_2, m}^{(6)} \right)^{n-1} L^{\frac{d_2}{d_1}(n-1)} \left(1 + \tau_g^{(d_2-m)(n-1)} \right)}{(n-1) \left(\frac{d_2}{d_1} - 1 \right)^{(n-1)} (n-1)!}, \quad (3.5)$$

where $C_{K_I, K_p, d_1, d_2, m}^{(6)}$ is a constant depending only on K_I, K_p, d_1, d_2, m .

Proof. in Appendix B. □

Next, Lemma 7 shows that the estimator \hat{d}_n in (3.4) is always correct when the intrinsic dimension is d_1 , as in (3.6). Specifically, the d_1 -squared length of the TSP path in (3.3) is bounded by some positive threshold $C_{K_I, K_v, d_1, m}^{(7)} (1 + \tau_g^{d_1-m})$. We take note that, when $d_1 = 1$, Lemma 7 is straightforward: the length of the TSP path in (3.3) is upper bounded by the length of curve $\text{vol}_M(M)$, as in Figure 3.1. This fact, combined with Lemma 3, which shows that $\text{vol}_M(M) \leq C_{K_I, d, m}^{(3,2)} (1 + \tau_g^{1-m})$, yields the result. In particular, the constant $C_{K_I, K_v, d_1, m}^{(7)}$ can be set as $C_{K_I, K_v, d_1, m}^{(7)} = C_{K_I, d, m}^{(3,2)}$.

When $d_1 > 1$, Lemma 7 is proved using Lemma 3, 4 and 5, along with the Hölder continuity of a d_1 -dimensional space-filling curve [Steele, 1997, Buchin, 2008].

Lemma 7. Fix $\tau_g, \tau_\ell \in (0, \infty]$, $K_I \in [1, \infty)$, $K_v \in (0, 2^{-m}]$, $d_1 \in \mathbb{N}$, with $\tau_g \leq \tau_\ell$. Let $M \in \mathcal{M}_{\tau_g, \tau_\ell, K_p, K_v}^{d_1}$ and $X_1, \dots, X_n \in M$. Then

$$\min_{\sigma \in S_n} \sum_{i=1}^{n-1} \|X_{\sigma(i+1)} - X_{\sigma(i)}\|_{\mathbb{R}^m}^{d_1} \leq C_{K_I, K_v, d_1, m}^{(7)} \left(1 + \tau_g^{d_1 - m}\right), \quad (3.6)$$

where $C_{K_I, K_v, d_1, m}^{(7)}$ is a constant depending only on m , d_1 , K_v , and K_I .

Proof. in Appendix B. □

Proposition 8 below is the main result of this section and follows directly from Lemma 6 and Lemma 7 above. Indeed, when the intrinsic dimension is d_2 , the risk of our estimator \hat{d}_n , is of order $O\left(n^{-\left(\frac{d_2}{d_1}-1\right)n}\right)$ by Lemma 6 and the union bound. On the other hand, when the intrinsic dimension is d_1 , the risk of our estimator \hat{d}_n is 0, because of Lemma 7. Since we assume that the intrinsic dimension is either d_1 or d_2 , the maximum risk of our estimator \hat{d}_n in (3.1) is of order $O\left(n^{-\left(\frac{d_2}{d_1}-1\right)n}\right)$, which also serves as an upper bound of the minimax risk R_n in (2.6).

Proposition 8. Fix $\tau_g, \tau_\ell \in (0, \infty]$, $K_I \in [1, \infty)$, $K_v \in (0, 2^{-m}]$, $K_p \in [(2K_I)^m, \infty)$, $d_1, d_2 \in \mathbb{N}$, with $\tau_g \leq \tau_\ell$ and $1 \leq d_1 < d_2 \leq m$. Then

$$\begin{aligned} & \inf_{\hat{d}_n} \sup_{P \in \mathcal{P}_1 \cup \mathcal{P}_2} \mathbb{E}_{P^{(n)}} \left[\ell \left(\hat{d}_n, d(P) \right) \right] \\ & \leq \left(C_{K_I, K_p, K_v, d_1, d_2, m}^{(8)} \right)^n \left(1 + \tau_g^{-\left(\frac{d_2}{d_1}m + m - 2d_2\right)n} \right) n^{-\left(\frac{d_2}{d_1}-1\right)n}, \end{aligned}$$

where $C_{K_I, K_p, K_v, d_1, d_2, m}^{(8)} \in (0, \infty)$ is a constant depending only on $K_I, K_p, K_v, d_1, d_2, m$ where

$$\mathcal{P}_1 = \mathcal{P}_{\tau_g, \tau_\ell, K_I, K_v, K_p}^{d_1}, \quad \mathcal{P}_2 = \mathcal{P}_{\tau_g, \tau_\ell, K_I, K_v, K_p}^{d_2}.$$

Proof. in Appendix B. □

4 Lower Bound for Choosing Between Two Dimensions

The goal of this section is to derive a lower bound for the minimax rate R_n . As in Section 3, we fix $d_1, d_2 \in \mathbb{N}$ with $1 \leq d_1 < d_2 \leq m$, and assume that the intrinsic dimension of data is either d_1 or d_2 as in (2.7). This assumption is relaxed in Section 5.

Our strategy is to find a subset $T \subset I^n \subset (\mathbb{R}^d)^n$ and two sets of distributions $\mathcal{P}_1^{d_1}$ and $\mathcal{P}_2^{d_2}$ with dimensions d_1 and d_2 , such that $\mathcal{P}_1^{d_1}$ and $\mathcal{P}_2^{d_2}$ satisfy the regularity conditions in Definition 2, and whenever the sample $X = (X_1, \dots, X_n)$ lies on T , one cannot easily distinguish whether the underlying distribution is from $\mathcal{P}_1^{d_1}$ or $\mathcal{P}_2^{d_2}$.

After constructing T , $\mathcal{P}_1^{d_1}$ and $\mathcal{P}_2^{d_2}$, we derive the lower bound using the following result, known as Le Cam's lemma.

Lemma 9. (Le Cam's Lemma) *Let \mathcal{P} be a set of probability measures on (Ω, \mathcal{F}) , and $\mathcal{P}_1, \mathcal{P}_2 \subset \mathcal{P}$ be such that for all $P \in \mathcal{P}_i$, $\theta(P) = \theta_i$ for $i = 1, 2$. For any $Q_i \in \text{co}(\mathcal{P}_i)$, let q_i be the density of Q_i with respect to a measure ν . Then*

$$\inf_{\hat{\theta}} \sup_{P \in \mathcal{P}} \mathbb{E}_P[\ell(\hat{\theta}, \theta(P))] \geq \frac{\Delta}{4} \int [q_1(x) \wedge q_2(x)] d\nu(x),$$

where $\Delta = \ell(\theta_1, \theta_2)$.

Proof. [See Yu, 1997, Chapter 29.2, Lemma 1]. □

Our construction for T , $\mathcal{P}_1^{d_1}$, and $\mathcal{P}_2^{d_2}$ is based on mimicking a space-filling curve. Intuitively, this gives the lower bound since it is difficult to differentiate a space-filling curve and a higher dimensional cube. In detail, we set

$$\begin{aligned} \mathcal{P}_1^{d_1} = \{ & \text{distributions supported on} \\ & \text{a space-filling-curve like } d_1\text{-dimensional manifold} \}, \end{aligned} \quad (4.1)$$

and

$$\mathcal{P}_2^{d_2} = \{ \text{uniform distributions on } [-K_I, K_I]^{d_2} \}. \quad (4.2)$$

To apply Le Cam's lemma, we construct a set $T \subset I^n$ so that, whenever $X = (X_1, \dots, X_n) \in T$, we cannot distinguish whether X is from $\mathcal{P}_1^{d_1}$ in (4.1) or $\mathcal{P}_2^{d_2}$ in (4.2). Then, for an appropriately chosen distribution Q_1 in the convex hull of $\mathcal{P}_1^{d_1}$ with density q_1 with respect to Lebesgue measure λ , and a density q_2 from the class $\mathcal{P}_2^{d_2}$, $\int_T [q_1(x) \wedge q_2(x)] d\lambda(x)$ is a lower bound on the minimax rate R_n in (2.6). Indeed, from Le Cam's Lemma 9, we have that

$$\begin{aligned} \inf_{\hat{\theta}} \sup_{P \in \mathcal{P}} \mathbb{E}_P[\ell(\hat{\theta}, \theta(P))] & \geq \frac{1}{4} \int [q_1(x) \wedge q_2(x)] d\lambda(x) \\ & \geq \frac{1}{4} \int_T [q_1(x) \wedge q_2(x)] d\lambda(x). \end{aligned} \quad (4.3)$$

For constructing the class $\mathcal{P}_1^{d_1}$ in (4.1), it will be sufficient to consider the case $d_1 = 1$. In fact, Lemma 10 states that the regularity conditions in Definition 2 are still preserved when the manifold M is a Cartesian product with a cube $[-K_I, K_I]^{\Delta^d}$. Hence

once we construct space-filling curves satisfying the required regularity conditions, we can form a Cartesian product with a cube to construct "space-filling" manifolds that still satisfy the same conditions.

Lemma 10. Fix $\tau_g, \tau_\ell \in (0, \infty]$, $K_I \in [1, \infty)$, $K_v \in (0, 2^{-m}]$, $d, \Delta d \in \mathbb{N}$, with $\tau_g \leq \tau_\ell$ and $1 \leq d + \Delta d \leq m$. Let $M \in \mathcal{M}_{\tau_g, \tau_\ell, K_I, K_v}^d$ be a d -dimensional manifold of global reach $\geq \tau_g$, local reach $\geq \tau_\ell$, which is embedded in $\mathbb{R}^{m-\Delta d}$. Then

$$M \times [-K_I, K_I]^{\Delta d} \in \mathcal{M}_{\tau_g, \tau_\ell, K_I, K_v}^{d+\Delta d},$$

which is embedded in \mathbb{R}^m .

Proof. in Appendix C. □

The precise construction of $\mathcal{P}_1^{d_1}$ in (4.1) and T is detailed in Lemma 11. As in Figure C.2, we construct T_i 's that are cylinder sets aligned as a zigzag in $[-K_I, K_I]^{d_2}$, and then T is constructed as $T = S_n \prod_{i=1}^n T_i$, where the permutation group S_n acts on $\prod_{i=1}^n T_i$ as a coordinate change. Then, we show below that, for any $x \in \prod T_i$, there exists a manifold $M \in \mathcal{M}_{\tau_g, \tau_\ell, \infty}^{d_2}$ that passes through x_1, \dots, x_n . The class $\mathcal{P}_1^{d_1}$ in (4.1) is finally defined as the set of distributions that are supported on such a manifold.

Lemma 11. Fix $\tau_\ell \in (0, \infty]$, $K_I \in [1, \infty)$, $d_1, d_2 \in \mathbb{N}$, with $1 \leq d_1 \leq d_2$, and suppose $2\tau_\ell \leq K_I$. Then there exist $T_1, \dots, T_n \subset [-K_I, K_I]^{d_2}$ such that:

- (1) The T_i 's are distinct.
- (2) For each T_i , there exists an isometry Φ_i such that

$$T_i = \Phi_i \left([-K_I, K_I]^{d_1-1} \times [0, a] \times B_{\mathbb{R}^{d_2-d_1}}(0, w) \right),$$

where $c = \left\lceil \frac{K_I + \tau_\ell}{2\tau_\ell} \right\rceil$, $a = \frac{K_I - \tau_\ell}{(d+\frac{1}{2}) \left\lceil \frac{n}{c^{d_2-d_1}} \right\rceil}$, and $w = \min \left\{ \tau_\ell, \frac{d^2(K_I - \tau_\ell)^2}{2\tau_\ell(d+\frac{1}{2})^2 \left(\left\lceil \frac{n}{c^{d_2-d_1}} \right\rceil + 1 \right)^2} \right\}$.

(3) There exists $\mathcal{M} : (B_{\mathbb{R}^{d_2-d_1}}(0, w))^n \rightarrow \mathcal{M}_{\tau_g, \tau_\ell, K_I, K_v}^{d_1}$ one-to-one such that for each $Y_i \in B_{\mathbb{R}^{d_2-d_1}}(0, w)$, $1 \leq i \leq n$, $\mathcal{M}(Y_1, \dots, Y_n) \cap T_i = \Phi_i([-K_I, K_I]^{d_1-1} \times [0, a] \times \{Y_i\})$. Hence for any $X_1 \in T_1, \dots, X_n \in T_n$, $\mathcal{M}(\{\Pi_{(d_1+1):d_2}^{-1} \Phi_i^{-1}(X_i)\}_{1 \leq i \leq n})$ passes through X_1, \dots, X_n .

Proof. in Appendix C. □

Next we show that whenever $X = (X_1, \dots, X_n) \in T$, it is difficult to tell whether the data originated from $P \in \mathcal{P}_1$ or $P \in \mathcal{P}_2$. From (4.3), we know that a lower bound is

given by $\int_T [q_1(x) \wedge q_2(x)] d\lambda(x)$. Hence if $q_1(x) \geq Cq_2(x)$ for every $x \in T$ with $C < 1$, then $q_1(x) \wedge q_2(x) \geq Cq_2(x)$, so that $C \int_T q_2(x)$ can serve as lower bound of minimax rate. The inequality $q_1(x) \geq Cq_2(x)$ is shown in Claim 12.

Claim 12. Let $T = S_n \prod_{i=1}^n T_i$ where the T_i 's are from Lemma 11, and let Q_1, Q_2 be from (C.15) in Proposition 13. Then for all $x \in \text{int}T$, there exists $r_x > 0$ such that for all $r < r_x$,

$$Q_1 \left(\prod_{i=1}^n B_{\|\cdot\|_{\mathbb{R}^{d_2}, \infty}}(x_i, r) \right) \geq \frac{\tau_\ell^{(d_2-d_1)n}}{2^n K_I^{(d_2-d_1)n}} Q_2 \left(\prod_{i=1}^n B_{\|\cdot\|_{\mathbb{R}^{d_2}, \infty}}(x_i, r) \right).$$

Proof. in Appendix C. □

The following lower bound is than a consequence of Le Cam's lemma, Lemma 11, and the previous claim.

Proposition 13. Fix $\tau_g, \tau_\ell \in (0, \infty]$, $K_I \in [1, \infty)$, $K_v \in (0, 2^{-m}]$, $K_p \in [(2K_I)^m, \infty)$, $d_1, d_2 \in \mathbb{N}$, with $\tau_g \leq \tau_\ell$ and $1 \leq d_1 < d_2 \leq m$, and suppose that $2\tau_\ell < K_I$. Then

$$\begin{aligned} & \inf_{\hat{d}} \sup_{P \in \mathcal{Q}} \mathbb{E}_{P^{(n)}} [\ell(\hat{d}_n, d(P))] \\ & \geq \left(C_{d_1, d_2, K_I}^{(13)} \right)^n \tau_\ell^{(d_2-d_1)n} \min \left\{ \tau_\ell^{-2(d_2-d_1)-1} n^{-2}, 1 \right\}^{(d_2-d_1)n}, \end{aligned}$$

where $C_{d_1, d_2, K_I}^{(13)} \in (0, \infty)$ is a constant depending only on d_1, d_2 , and K_I and

$$\mathcal{Q} = \mathcal{P}_{\tau_g, \tau_\ell, K_I, K_v, K_p}^{d_1} \bigcup \mathcal{P}_{\tau_g, \tau_\ell, K_I, K_v, K_p}^{d_2}.$$

Proof. in Appendix C. □

5 Upper Bound and Lower Bound for the General Case

Now we generalize our results to allow the intrinsic dimension d to be any integer between 1 and m . Thus the model is $\mathcal{P} = \bigcup_{d=1}^m \mathcal{P}_{\tau_g, \tau_\ell, K_I, K_v, K_p}^d$ as in (2.8). The estimator \hat{d} that we consider to derive the upper bound is the smallest integer $1 \leq d \leq m$ such that (3.6) holds. As for the lower bound, we simply use the lower bound derived in Section 4 with $d_1 = 1$ and $d_2 = 2$.

Proposition 14. Fix $\tau_g, \tau_\ell \in (0, \infty]$, $K_I \in [1, \infty)$, $K_v \in (0, 2^{-m}]$, $K_p \in [(2K_I)^m, \infty)$, $d_1, d_2 \in \mathbb{N}$, with $\tau_g \leq \tau_\ell$ and $1 \leq d_1 < d_2 \leq m$. Then:

$$\inf_{\hat{d}_n} \sup_{P \in \mathcal{P}^{(n)}} \left[\ell(\hat{d}_n, d(P)) \right] \leq \left(C_{K_I, K_p, K_v, m}^{(14)} \right)^n \left(1 + \tau_g^{-(m^2-m)n} \right) n^{-\frac{1}{m-1}n}$$

where $C_{K_I, K_p, K_v, m}^{(14)} \in (0, \infty)$ is a constant depending only on K_I, K_p, K_v, m .

Proof. in Appendix D. □

Proposition 15 provides a lower bound for minimax rate R_n in (2.6), in multi-dimensions. It can be viewed of a generalization for the binary dimension case in Proposition 13.

Proposition 15. Fix $\tau_g, \tau_\ell \in (0, \infty]$, $K_I \in [1, \infty)$, $K_v \in (0, 2^{-m}]$, $K_p \in [(2K_I)^m, \infty)$, $d_1, d_2 \in \mathbb{N}$, with $\tau_g \leq \tau_\ell$ and $1 \leq d_1 < d_2 \leq m$, and suppose that $\tau_\ell < K_I$. Then,

$$\inf_{\hat{d}} \sup_{P \in \mathcal{P}} \mathbb{E}_{P^{(n)}}[\ell(\hat{d}_n, d(P))] \geq \left(C_{K_I}^{(15)}\right)^n \tau_\ell^n \min\{\tau_\ell^{-3} n^{-2}, 1\}^n$$

where $C_{K_I}^{(15)} \in (0, \infty)$ is a constant depending only on K_I .

Proof. in Appendix D. □

6 Conclusion

On a logarithmic scale, the leading terms of the lower and upper bounds for the minimax rate R_n in (2.6) have the form

$$-nc \log \tau$$

for some constant c , where τ is the global reach for the upper bound and the local reach for the lower bound. This shows that the difficulty of the problem of estimating the dimension goes to 0 rapidly with sample size, in a way that depends on the curvature of the manifold.

There are several open problems. The first is to tighten the bounds so that the upper and lower bounds match. Second, it should be possible to extend the analysis to allow noise. With enough noise, the minimax rate should eventually become the same as the rate in [Koltchinskii, 2000]. Finally, it would be interesting to get very precise bounds on the many dimension estimators that appear in the literature and compare these bounds to the minimax bounds.

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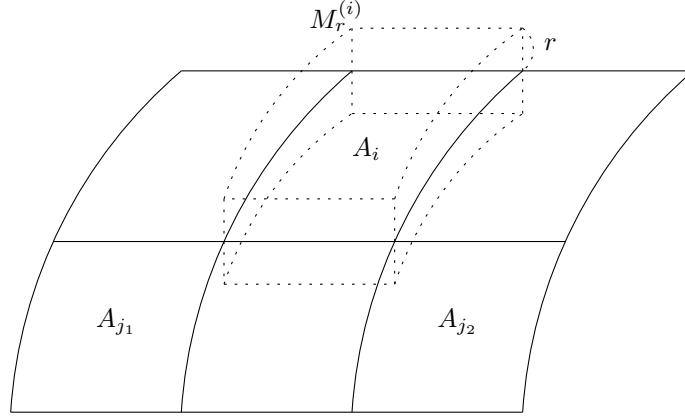


Figure A.1: $\{A_1, \dots, A_l\}$ is a disjoint cover of M , and each A_i is a projection of $M_r^{(i)}$ on M .

A Proofs for Section 2

Lemma 3. Fix $\tau_g, \tau_\ell \in (0, \infty]$, $K_I \in [1, \infty)$, $K_v \in (0, 2^{-m}]$, with $\tau_g \leq \tau_\ell$. For $M \in \mathcal{M}_{\tau_g, \tau_\ell, K_I, K_v}^d$ and $r \in (0, \tau_g)$, let $M_r := \{x \in \mathbb{R}^m : \text{dist}_{\mathbb{R}^m}(x, M) < r\}$ be a r -neighborhood of M in \mathbb{R}^m . Then, the volume of M is upper bounded as

$$\begin{aligned} \text{vol}_M(M) &\leq C_{d,m}^{(3,1)} r^{d-m} \text{vol}_{\mathbb{R}^m}(M_r) \\ &\leq C_{K_I, d, m}^{(3,2)} \left(1 + \tau_g^{d-m}\right), \end{aligned} \quad (\text{A.1})$$

where $C_{d,m}^{(3,1)}$ is a constant depending only on d and m , and $C_{K_I, d, m}^{(3,2)}$ is a constant depending only on K_I , d and m .

Proof. Suppose $\{A_1, \dots, A_l\}$ is a disjoint cover of M , i.e. measurable subsets of M such that $A_i \cap A_j = \emptyset$, $\bigcup_{i=1}^l A_i = M$, and each A_i is equipped with chart maps $\varphi^{(i)} : U_i \subset \mathbb{R}^d \rightarrow A_i$.

Such a triangulation is always possible. For each A_i , define $M_r^{(i)} := \{x \in \mathbb{R}^m : \pi_M(x) \in A_i, \text{dist}_{\mathbb{R}^m, \|\cdot\|_1}(x, M) \leq r\}$ so that each A_i is a projection of $M_r^{(i)}$ on M , as in Figure A.1. Then,

$$\text{vol}_{\mathbb{R}^m}(M_r) = \sum_{i=1}^l \text{vol}_{\mathbb{R}^m}(M_r^{(i)}). \quad (\text{A.2})$$

Fix $i \in \{1, \dots, l\}$. Then for each $u \in U_i$, there exists a linear isometry $R^{(i)}(u) : \mathbb{R}^{m-d} \rightarrow (T_{\varphi^{(i)}(u)}M)^\perp$, which can be identified as an $m \times (m-d)$ matrix with j^{th} column being

$R^{(i,j)}(u)$, so that $M_r^{(i)}$ can be parametrized as $\psi^{(i)} : U_i \times B_{\|\cdot\|_{\mathbb{R}^{m-d},1}}(0, r) \rightarrow M_r^{(i)}$ with

$$\psi^{(i)}(u, t) = \varphi^{(i)}(u) + R^{(i)}(u)t = \varphi^{(i)}(u) + \sum_{j=1}^{m-d} t_j R^{(i,j)}(u). \quad (\text{A.3})$$

Then, because $R^{(i)}$ is an isometry,

$$R^{(i)}(u)^\top R^{(i)}(u) = I_{m-d}. \quad (\text{A.4})$$

Let $\psi_u^{(i)} = \frac{\partial \psi^{(i)}}{\partial u} = \left(\frac{\partial \psi^{(i)}}{\partial u_1}, \dots, \frac{\partial \psi^{(i)}}{\partial u_d} \right) \in \mathbb{R}^{m \times d}$ be the partial derivative of $\psi^{(i)}$ with respect to u and let $\psi_t^{(i)} = \frac{\partial \psi^{(i)}}{\partial t}$ be the partial derivative of $\psi^{(i)}$ with respect to t . Define $\varphi_u^{(i)}$, $\varphi_t^{(i)}$, $R_u^{(i,j)}$, $R_t^{(i,j)}$ similarly. Then, since $R^{(i)}$ is an isometry, $\text{image}(R^{(i)}(u)) = (T_{\varphi^{(i)}(u)} M)^\perp$ holds, and hence

$$R^{(i)}(u)^\top \varphi_u^{(i)}(u) = 0. \quad (\text{A.5})$$

Also by differentiating (A.4), for all j ,

$$R_u^{(i,j)}(u)^\top R^{(i)}(u) = 0. \quad (\text{A.6})$$

Also by differentiating (A.3), we get

$$\psi_u^{(i)}(u, t) = \varphi_u^{(i)}(u) + \sum_{j=1}^{m-d} t_j R_u^{(i,j)}(u), \quad (\text{A.7})$$

and

$$\psi_t^{(i)}(u, t) = R^{(i)}(u). \quad (\text{A.8})$$

Hence by multiplying (A.7) and (A.8), and by applying (A.4), (A.5), and (A.6), we get

$$\psi_t^{(i)}(u, t)^\top \psi_u^{(i)}(u, t) = R^{(i)}(u)^\top \varphi_u^{(i)}(u) + R^{(i)}(u)^\top R_u^{(i)}(u)t = 0, \quad (\text{A.9})$$

and

$$\psi_t^{(i)}(u, t)^\top \psi_t^{(i)}(u, t) = R^{(i)}(u)^\top R^{(i)}(u) = I_{m-d}. \quad (\text{A.10})$$

Now let's consider $\psi_u^{(i)}(u, t)^\top \psi_u^{(i)}(u, t)$. From (A.6) and $\text{image}(R^{(i)}(u)) = (T_{\varphi^{(i)}(u)} M)^\perp$, column space generated by $R_u^{(i,j)}(u)$ is contained in $T_{\varphi^{(i)}(u)} M$, i.e.

$$\langle R_u^{(i,j)}(u) \rangle \subset T_{\varphi^{(i)}(u)}(M) = \text{span}(\varphi_u^{(i)}(u)).$$

Therefore, there exists $\Lambda^{(i,j)}(u) : d \times d$ matrix such that

$$R_u^{(i,j)}(u) = \varphi_u^{(i)}(u) \Lambda^{(i,j)}(u).$$

Then by applying this to (A.7),

$$\psi_u^{(i)}(u, t) = \varphi_u^{(i)}(u) \left(I + \sum_{j=1}^{m-d} t_j \Lambda^{(i,j)}(u) \right). \quad (\text{A.11})$$

Now M being of global reach $\geq \tau_g$ implies $\psi_u^{(i)}(u, t)$ is of full rank for all $t \in B_{\mathbb{R}^{m-d}}(0, \tau_g)$. Hence from (A.11), this implies that $I + \sum_{j=1}^{m-d} t_j \Lambda^{(i,j)}(u)$ is invertible for all $t \in B_{\mathbb{R}^{m-d}}(0, \tau_g)$, and this implies all singular values of $\Lambda^{(i,j)}(u)$ is bounded by κ_g . Hence for all $v \in \mathbb{R}^d$,

$$\left| v^\top \Lambda^{(i,j)}(u) v \right| \leq \kappa_g \|v\|_2^2,$$

and accordingly,

$$\begin{aligned} \left| v^\top \left(I + \sum_{j=1}^{m-d} t_j \Lambda^{(i,j)}(u) \right) v \right| &\geq \|v\|_2^2 - \sum_{j=1}^{m-d} |t_j| \left| v^\top \Lambda^{(i,j)}(u) v \right| \\ &\geq (1 - \|t\|_1 \kappa_g) \|v\|_2^2. \end{aligned}$$

Hence any singular values σ of $I + \sum_{j=1}^{m-d} t_j \Lambda^{(i,j)}(u)$ satisfies $|\sigma| \geq 1 - \|t\|_1 \kappa_g$. And since $\|t\|_1 \leq \tau_g$,

$$\left| I + \sum_{j=1}^{m-d} t_j \Lambda^{(i,j)}(u) \right| \geq (1 - \|t\|_1 \kappa_g)^d.$$

By applying this result to (A.11), the determinant of $\psi_u^{(i)}(u, t)^\top \psi_u^{(i)}(u, t)$ is lower bounded as

$$\begin{aligned} \left| \psi_u^{(i)}(u, t)^\top \psi_u^{(i)}(u, t) \right| &= \left| I + \sum_{j=1}^{m-d} t_j \Lambda^{(i,j)}(u) \right|^2 \left| \varphi_u^{(i)}(u)^\top \varphi_u^{(i)}(u) \right| \\ &\geq (1 - \|t\|_1 \kappa_g)^{2d} \left| \varphi_u^{(i)}(u)^\top \varphi_u^{(i)}(u) \right|. \end{aligned} \quad (\text{A.12})$$

Now, let $g_{ij}^{(M_r)}$ be the Riemannian metric tensor of M_r , and $g_{ij}^{(M)}$ be the Riemannian metric tensor of M . Then from (A.9), (A.10), and (A.12), the determinant of Riemannian metric

tensor $g_{ij}^{(M_r)}$ is lower bounded by

$$\begin{aligned}
|\det(g_{ij}^{(M_r)})|^2 &= \left| \left(\psi_u^{(i)}(u, t) \ \psi_t^{(i)}(u, t) \right)^\top \left(\psi_u^{(i)}(u, t) \ \psi_t^{(i)}(u, t) \right) \right| \\
&= \left| \begin{array}{cc} \psi_u^{(i)}(u, t)^\top \psi_u^{(i)}(u, t) & \psi_u^{(i)}(u, t)^\top \psi_t^{(i)}(u, t) \\ \psi_u^{(i)}(u, t)^\top \psi_t^{(i)}(u, t) & \psi_t^{(i)}(u, t)^\top \psi_t^{(i)}(u, t) \end{array} \right| \\
&= \left| \psi_u^{(i)}(u, t)^\top \psi_u^{(i)}(u, t) \right| \\
&\geq (1 - \|t\|_1 \kappa_g)^{2d} \left| \varphi_u^{(i)}(u)^\top \varphi_u^{(i)}(u) \right| \\
&= (1 - \|t\|_1 \kappa_g)^{2d} |\det(g_{ij}^{(M)})|^2.
\end{aligned}$$

And from this, volume of $M_r^{(i)}$ is lower bounded as

$$\begin{aligned}
\text{vol}_{\mathbb{R}^m}(M_r^{(i)}) &= \int_{U_i \times B_{\mathbb{R}^m, \|\cdot\|_1}(0, r)} |\det(g_{ij}^{(M_r)})| du dt \\
&\geq \int_{U_i} \int_{B_{\mathbb{R}^m, \|\cdot\|_1}(0, r)} (1 - \|t\|_1 \kappa_g)^d |\det(g_{ij}^{(M)})| dt du \\
&= \text{vol}(U_i) \int_0^r \int_{t_1 + \dots + t_{m-d-1} \leq s} (1 - s \kappa_g)^d dt_1 \dots dt_{m-d-1} ds \\
&= \frac{1}{(m-d-1)!} \text{vol}(U_i) \int_0^r s^{m-d-1} (1 - s \kappa_g)^d ds \\
&\geq \frac{1}{C_{d,m}^{(3,1)}} r^{m-d} \text{vol}(U_i), \tag{A.13}
\end{aligned}$$

where $C_{d,m}^{(3,1)} \in (0, \infty)$ is a constant depending only on d and m . By applying (A.13) to (A.2), we can lower bound volume of M_r as

$$\begin{aligned}
\text{vol}_{\mathbb{R}^m}(M_r) &\geq \frac{1}{C_{d,m}^{(3,1)}} r^{m-d} \sum_{i=1}^l \text{vol}(U_i) \\
&= \frac{1}{C_{d,m}^{(3,1)}} r^{m-d} \text{vol}_M(M). \tag{A.14}
\end{aligned}$$

Also, with $r = \tau_g$, M_r is contained in τ_g -neighborhood of I , hence

$$\text{vol}_{\mathbb{R}^m}(M_r) \leq 2^m (K_I + \tau_g)^m. \tag{A.15}$$

By combining (A.14) and (A.15), we get the desired upper bound of $\text{vol}_M(M)$ in (A.1) as

$$\begin{aligned}
\text{vol}_M(M) &\leq C_{d,m}^{(3,1)} r^{d-m} \text{vol}_{\mathbb{R}^m}(M_r) \\
&\leq C_{K_I, d, m}^{(3,2)} \left(1 + \tau_g^{d-m} \right),
\end{aligned}$$

where $C_{K_I, d, m}^{(3,2)} \in (0, \infty)$ is a constant depending only on K_I , d and m . □

Lemma 16. (*Toponogov comparison theorem, 1959*) Let (M, g) be a complete Riemannian manifold with sectional curvature $\geq \kappa$, and let S_κ be a surface of constant Gaussian curvature κ . Given any geodesic triangle with vertices $p, q, r \in M$ forming an angle α at q , consider a (comparison) triangle with vertices $\bar{p}, \bar{q}, \bar{r} \in S_\kappa$ such that $\text{dist}_{S_\kappa}(\bar{p}, \bar{q}) = \text{dist}_M(p, q)$, $\text{dist}_{S_\kappa}(\bar{r}, \bar{q}) = \text{dist}_M(r, q)$, and $\angle \bar{p}\bar{q}\bar{r} = \angle pqr$. Then

$$\text{dist}_M(\bar{p}, \bar{r}) \leq \text{dist}_{S_\kappa}(p, r).$$

Proof. [See Petersen, 2006, Theorem 79, p.339]. Note that for a manifold with boundary, the complete Riemannian manifold condition can be relaxed to requiring the existence of a geodesic path joining p and q whose image lies on $\text{int}M$. □

Lemma 17. (*Hyperbolic law of cosines*) Let H_κ be a hyperbolic plane whose Gaussian curvature is $-\kappa^2$. Then given a hyperbolic triangle ABC with angles α, β, γ , and side lengths $BC = a$, $CA = b$, and $AB = c$, the following holds:

$$\cosh(\kappa a) = \cosh(\kappa b) \cosh(\kappa c) - \sinh(\kappa b) \sinh(\kappa c) \cos \alpha.$$

Proof. [See Bridson and Häflicher, 1999, 2.13 The Law of Cosines in M_κ^n , p.24]. □

Claim 18. Let $\lambda \in [0, 1]$ and let $a, b \in [0, \infty)$ satisfy $a < b$. Then

$$\frac{\cosh^{-1}((1-\lambda)\cosh a + \lambda\cosh b)}{\sqrt{(1-\lambda)a^2 + \lambda b^2}} \leq \frac{\sinh(\frac{b}{2})}{b/2}. \quad (\text{A.16})$$

Proof. Consider functions $F, G : [0, \infty)^2 \times [0, 1] \rightarrow \mathbb{R}$ defined as $F(a, b, \lambda) = f^{-1}((1-\lambda)f(a) + \lambda f(b))$ and $G(a, b, \lambda) = g^{-1}((1-\lambda)g(a) + \lambda g(b))$, for $0 \leq a < b$, $\lambda \in [0, 1]$, $f(t) = \cosh t$, and $g(t) = t^2$. Toponogov comparison theorem in Lemma 16 implies $F(a, b, \lambda) \geq G(a, b, \lambda)$, and f and g being strictly increasing function implies $a < G(a, b, \lambda) \leq F(a, b, \lambda) < b$. Also differentiating log fraction $\frac{\partial}{\partial a} \log \frac{F(a, b, \lambda)}{G(a, b, \lambda)}$ gives

$$\begin{aligned} \frac{\partial}{\partial a} \log \frac{F(a, b, \lambda)}{G(a, b, \lambda)} &= \frac{(1-\lambda)f'(a)}{f'(F(a, b, \lambda))F(a, b, \lambda)} - \frac{(1-\lambda)g'(a)}{g'(G(a, b, \lambda))G(a, b, \lambda)} \\ &= \frac{1-\lambda}{F(a, b, \lambda)} \exp\left(-\int_a^{F(a, b, \lambda)} (\log f')'(t) dt\right) \\ &\quad - \frac{1-\lambda}{G(a, b, \lambda)} \exp\left(-\int_a^{G(a, b, \lambda)} (\log g')'(t) dt\right). \end{aligned} \quad (\text{A.17})$$

Then by applying $(\log f')'(t) = \coth t > \frac{1}{t} = (\log g')'(t)$ and $F(a, b, \lambda) \geq G(a, b, \lambda)$ to (A.17) implies

$$0 < \forall a < b, \quad \frac{\partial}{\partial a} \log \frac{F(a, b, \lambda)}{G(a, b, \lambda)} < 0,$$

and hence

$$\frac{F(a, b, \lambda)}{G(a, b, \lambda)} \leq \frac{F(0, b, \lambda)}{G(0, b, \lambda)}.$$

By expanding F and G from this, we get

$$\begin{aligned} \frac{\cosh^{-1}((1-\lambda)\cosh a + \lambda\cosh b)}{\sqrt{(1-\lambda)a^2 + \lambda b^2}} &\leq \frac{\cosh^{-1}(\lambda\cosh b + (1-\lambda))}{\sqrt{\lambda b^2}} \\ &= \frac{\cosh^{-1}(1 + 2\lambda \sinh^2(\frac{b}{2}))}{b\sqrt{\lambda}} \\ &\leq \frac{2\sinh(\frac{b}{2})}{b}, \end{aligned}$$

where last line is coming from $1 + x \leq \cosh \sqrt{2x} \implies \cosh^{-1}(1 + x) \leq \sqrt{2x}$. Hence we get (A.16). □

Lemma 5. Fix $\tau_g, \tau_\ell \in (0, \infty]$, $K_I \in [1, \infty)$, $K_v \in (0, 2^{-m}]$, with $\tau_g \leq \tau_\ell$. Let $M \in \mathcal{M}_{\tau_g, \tau_\ell, K_I, K_v}^d$ and let $\exp_{p_k} : \mathcal{E}_k \subset \mathbb{R}^m \rightarrow \mathcal{M}$ be an exponential map, where \mathcal{E}_k is the domain of the exponential map \exp_{p_k} and $T_{p_k}M$ is identified with \mathbb{R}^m . For all $v, w \in \mathcal{E}_k$, let $R_k := \max\{\|v\|, \|w\|\}$. Then

$$\|\exp_{p_k}(v) - \exp_{p_k}(w)\|_{\mathbb{R}^m} \leq \frac{\sinh(\kappa_\ell R_k)}{\kappa_\ell R_k} \|v - w\|_{\mathbb{R}^d}. \quad (\text{A.18})$$

Proof. Let $q_1 = \exp_{p_k}(v)$ and $q_2 = \exp_{p_k}(w)$. Let $\text{dist}_M(p_k, q_1) = \tau_\ell r_1$, $\text{dist}_M(p_k, q_2) = \tau_\ell r_2$, and $\angle q_1 p_k q_2 = 2\alpha$ with $0 \leq \alpha \leq \pi$, as in Figure A.2(a). Then

$$\begin{aligned} \|v - w\|_{\mathbb{R}^{d_1}} &= \tau_\ell \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos 2\alpha} \\ &= \tau_\ell \sqrt{(r_1 + r_2)^2 \sin^2 \alpha + (r_1 - r_2)^2 \cos^2 \alpha}. \end{aligned} \quad (\text{A.19})$$

Let H_{κ_l} be a surface of constant sectional curvature $-\kappa_l^2$, and let $\bar{p}_k, \bar{q}_1, \bar{q}_2 \in H_{\kappa_l}$ be such that $\text{dist}_{H_{\kappa_l}}(\bar{p}_k, \bar{q}_1) = \text{dist}_M(p_k, q_1)$, $\text{dist}_{H_{\kappa_l}}(\bar{p}_k, \bar{q}_2) = \text{dist}_M(p_k, q_2)$, and $\angle \bar{q}_1 \bar{p}_k \bar{q}_2 = \angle q_1 p_k q_2$, so that $\triangle \bar{p}_k \bar{q}_1 \bar{q}_2$ becomes a comparison triangle of $p_k q_1 q_2$, as in Figure A.2(b). Then since

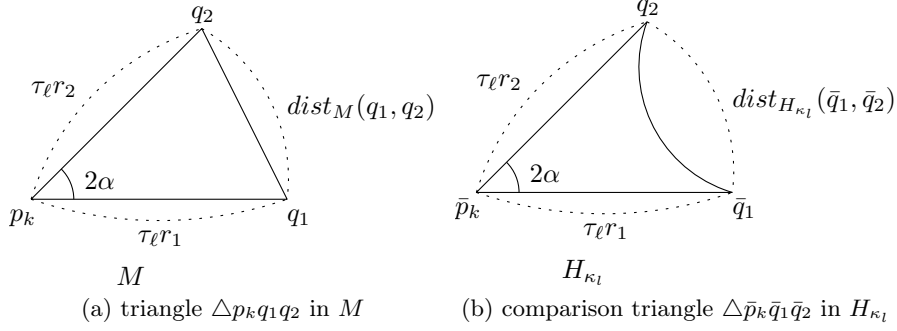


Figure A.2: (a) triangle $\triangle p_k q_1 q_2$ in M formed by p_k , q_1 , q_2 , and (b) its comparison triangle $\triangle \bar{p}_k \bar{q}_1 \bar{q}_2$ in H_{κ_l} .

(sectional curvature of M) $\geq -\kappa_l^2$, from the Toponogov comparison theorem in Lemma 16,

$$\text{dist}_M(q_1, q_2) \leq \text{dist}_{H_{\kappa_l}}(\bar{q}_1, \bar{q}_2). \quad (\text{A.20})$$

Also, by applying the hyperbolic law of cosines in Lemma 17 to comparison triangle $\triangle \bar{p}_k \bar{q}_1 \bar{q}_2$ in Figure A.2(a),

$$\begin{aligned} \cosh(\kappa_l \text{dist}_{H_{\kappa_l}}(\bar{q}_1, \bar{q}_2)) &= \cosh r_1 \cosh r_2 - \sinh r_1 \sinh r_2 \cos 2\alpha \\ &= (\sin^2 \alpha) \cosh(r_1 + r_2) + (\cos^2 \alpha) \cosh(r_1 - r_2). \end{aligned} \quad (\text{A.21})$$

From (A.19) and (A.21), we can expand the fraction of distances $\frac{\text{dist}_{H_{\kappa_l}}(\bar{q}_1, \bar{q}_2)}{\|v-w\|_{\mathbb{R}^d}}$ as

$$\frac{\text{dist}_{H_{\kappa_l}}(\bar{q}_1, \bar{q}_2)}{\|v-w\|_{\mathbb{R}^d}} = \frac{\cosh^{-1}(\sin^2 \alpha \cosh(r_1 + r_2) + \cos^2 \alpha \cosh(r_1 - r_2))}{\sqrt{(\sin^2 \alpha)(r_1 + r_2)^2 + (\cos^2 \alpha)(r_1 - r_2)^2}}. \quad (\text{A.22})$$

Then we can upper bound the fraction of distances $\frac{\text{dist}_{H_{\kappa_l}}(\bar{q}_1, \bar{q}_2)}{\|v-w\|_{\mathbb{R}^d}}$ by plugging in $a = |r_1 - r_2|$, $b = r_1 + r_2$, $\lambda = \sin^2 \alpha$ to Claim 18 implies

$$\frac{\cosh^{-1}(\sin^2 \alpha \cosh(r_1 + r_2) + \cos^2 \alpha \cosh(r_1 - r_2))}{\sqrt{(\sin^2 \alpha)(r_1 + r_2)^2 + (\cos^2 \alpha)(r_1 - r_2)^2}} \leq \frac{\sinh\left(\frac{r_1 + r_2}{2}\right)}{(r_1 + r_2)/2}. \quad (\text{A.23})$$

Then since $t \mapsto \frac{\sinh t}{t}$ is an increasing function of t and $\frac{r_1 + r_2}{2} \leq \kappa_\ell R_k$, so

$$\frac{\sinh\left(\frac{r_1 + r_2}{2}\right)}{(r_1 + r_2)/2} \leq \frac{\sinh(\kappa_\ell R_k)}{\kappa_\ell R_k}. \quad (\text{A.24})$$

Combining (A.22), (A.23), and (A.24), we have upper bound of the fraction of distances $\frac{dist_{H_{\kappa_l}}(\bar{q}_1, \bar{q}_2)}{\|v-w\|_{\mathbb{R}^d}}$ uniform over r_1, r_2 as

$$\frac{dist_{H_{\kappa_l}}(\bar{q}_1, \bar{q}_2)}{\|v-w\|_{\mathbb{R}^d}} \leq \frac{\sinh(\kappa_\ell R_k)}{\kappa_\ell R_k}. \quad (\text{A.25})$$

And finally, combining (A.20) and (A.25), we get desired upper bound of $\|\exp_{p_k}(v) - \exp_{p_k}(w)\|_{\mathbb{R}^m}$ in (A.18) as

$$\begin{aligned} \|\exp_{p_k}(v) - \exp_{p_k}(w)\|_{\mathbb{R}^m} &\leq dist_M(q_1, q_2) \\ &\leq dist_{H_{\kappa_l}}(\bar{q}_1, \bar{q}_2) \\ &\leq \frac{\sinh(\kappa_\ell R_k)}{\kappa_\ell R_k} \|v-w\|_{\mathbb{R}^d}. \end{aligned}$$

□

B Proofs for Section 3

Claim 19. Fix $\tau_g, \tau_\ell \in (0, \infty]$, $K_I \in [1, \infty)$, $K_v \in (0, 2^{-m}]$, $K_p \in [(2K_I)^m, \infty)$, $d_1, d_2 \in \mathbb{N}$, with $\tau_g \leq \tau_\ell$ and $1 \leq d_1 < d_2 \leq m$. Let $X_1, \dots, X_n \sim P \in \mathcal{P}_{\tau_g, \tau_\ell, K_I, K_v, K_p}^{d_2}$. Then for all $y \in [0, \infty)$,

$$P^{(n)}\left(\|X_n - X_{n-1}\|_{\mathbb{R}^m}^{d_1} \leq y | X_1, \dots, X_{n-1}\right) \leq C_{K_I, K_p, d_2, m}^{(19)} \left(1 + \tau_g^{d_2-m}\right) y^{\frac{d_2}{d_1}}, \quad (\text{B.1})$$

where $C_{K_I, K_p, d_2, m}^{(19)}$ is a constant depending only on K_I, K_p, d_2, m .

Proof. Let p_{X_n} be the pdf of X_n . Then conditional cdf of $\|X_n - X_{n-1}\|_{\mathbb{R}^m}^{d_1}$ given X_1, \dots, X_{n-1} is upper bounded by volume of a ball in the manifold M as

$$\begin{aligned} &P^{(n)}\left(\|X_n - X_{n-1}\|_{\mathbb{R}^m}^{d_1} \leq y | X_1, \dots, X_{n-1}\right) \\ &= P^{(n)}\left(X_n \in B_{\mathbb{R}^m}\left(X_{n-1}, y^{\frac{1}{d_1}}\right) | X_1, \dots, X_{n-1}\right) \\ &= \int_{M \cap \left(B_{\mathbb{R}^m}\left(X_{n-1}, y^{\frac{1}{d_1}}\right)\right)} p_{X_n}(x_n) d\text{vol}_M(x_n) \\ &\leq K_p \text{vol}_M\left(M \cap B\left(X_{n-1}, y^{\frac{1}{d_1}}\right)\right), \end{aligned} \quad (\text{B.2})$$

where last inequality is coming from condition (6) in Definition 2. And by applying Lemma 3, $\text{vol}_M \left(M \cap B \left(X_{n-1}, y^{\frac{1}{d_1}} \right) \right)$ can be further bounded as

$$\begin{aligned}
& \text{vol}_M \left(M \cap B \left(X_{n-1}, y^{\frac{1}{d_1}} \right) \right) \\
& \leq C_{d_2, m}^{(3,1)} \min \left\{ y^{\frac{1}{d_1}}, \tau_g \right\}^{d_2 - m} \text{vol}_{\mathbb{R}^m} \left(B \left(X_{n-1}, y^{\frac{1}{d_1}} + \min \left\{ y^{\frac{1}{d_1}}, \tau_g \right\} \right) \right) \quad (\text{Lemma 3}) \\
& = C_{d_2, m}^{(3,1)} \omega_m \left(y^{\frac{d_2}{d_1}} 2^m 1(y^{\frac{1}{d_1}} \leq \tau_g) + y^{\frac{d_2}{d_1}} \left(\frac{\tau_g}{y^{\frac{1}{d_1}}} \right)^{d_2 - m} \left(1 + \left(\frac{\tau_g}{y^{\frac{1}{d_1}}} \right)^m 1(y^{\frac{1}{d_1}} > \tau_g) \right) \right) \\
& \leq C_{d_2, m}^{(3,1)} \omega_m 2^m \left(y^{\frac{d_2}{d_1}} 1(y^{\frac{1}{d_1}} \leq \tau_g) + y^{\frac{d_2}{d_1}} \left(\frac{\tau_g}{2K_I \sqrt{m}} \right)^{d_2 - m} 1(y^{\frac{1}{d_1}} > \tau_g) \right) \\
& \leq C_{K_I, K_p, d_2, m}^{(3,1,1)} \left(1 + \tau_g^{d_2 - m} \right) y^{\frac{d_2}{d_1}} \quad (\text{B.3})
\end{aligned}$$

where $C_{K_I, d_2, m}^{(19,1)} = C_{d_2, m}^{(3,1)} \omega_m 2^m (2K_I \sqrt{m})^{m - d_2}$. By applying (B.2) and (B.3), we get the upper bound on conditional cdf of $\|X_n - X_{n-1}\|_{\mathbb{R}^m}^{d_1}$ given X_1, \dots, X_{n-1} in (B.1) as

$$\begin{aligned}
P^{(n)} \left(\|X_n - X_{n-1}\|_{\mathbb{R}^m}^{d_1} \leq y | X_1, \dots, X_{n-1} \right) & \leq K_p C_{K_I, d_2, m}^{(19,1)} \left(1 + \tau_g^{d_2 - m} \right) y^{\frac{d_2}{d_1}} \\
& \leq C_{K_I, K_p, d_2, m}^{(19)} \left(1 + \tau_g^{d_2 - m} \right) y^{\frac{d_2}{d_1}}, \quad (\text{B.4})
\end{aligned}$$

where $C_{K_I, K_p, d_2, m}^{(19)} = K_p C_{K_I, d_2, m}^{(19,1)} = K_p C_{d_2, m}^{(3,1)} \omega_m 2^m (2K_I \sqrt{m})^{m - d_2}$. □

Lemma 6. Fix $\tau_g, \tau_\ell \in (0, \infty]$, $K_I \in [1, \infty)$, $K_v \in (0, 2^{-m}]$, $K_p \in [(2K_I)^m, \infty)$, $d_1, d_2 \in \mathbb{N}$, with $\tau_g \leq \tau_\ell$ and $1 \leq d_1 < d_2 \leq m$. Let $X_1, \dots, X_n \sim P \in \mathcal{P}_{\tau_g, \tau_\ell, K_I, K_v, K_p}^{d_2}$. Then for all $L > 0$,

$$P^{(n)} \left[\sum_{i=1}^{n-1} \|X_{i+1} - X_i\|^{d_1} \leq L \right] \leq \frac{\left(C_{K_I, K_p, d_1, d_2, m}^{(6)} \right)^{n-1} L^{\frac{d_2}{d_1}(n-1)} \left(1 + \tau_g^{(d_2 - m)(n-1)} \right)}{(n-1)^{\left(\frac{d_2}{d_1} - 1 \right)(n-1)} (n-1)!}, \quad (\text{B.5})$$

where $C_{K_I, K_p, d_1, d_2, m}^{(6)}$ is a constant depending only on K_I, K_p, d_1, d_2, m .

Proof. Let $Y_i := \|X_{i+1} - X_i\|_{\mathbb{R}^m}^{d_1}$, $i = 1, \dots, n-1$. Then from Claim 19, probability of d_1 -squared length of the path being bounded by L , $P^{(n)} \left(\sum_{i=1}^{n-1} Y_i \leq L \right)$, is upper bounded

as

$$\begin{aligned}
& P^{(n)} \left(\sum_{i=1}^{n-1} Y_i \leq L \right) \\
&= \int_0^L P^{(n)} \left(Y_{n-1} \leq y_{n-1} \mid \sum_{i=1}^{n-2} Y_i = L - y_{n-1} \right) dF_{\sum_{i=1}^{n-2} Y_i}^{n-2}(L - y_{n-1}) \\
&\leq C_{K_I, K_p, d_2, m}^{(19)} \left(1 + \tau_g^{d_2-m} \right) \int_0^L y_{n-1}^{\frac{d_2}{d_1}} dF_{\sum_{i=1}^{n-2} Y_i}^{n-2}(L - y_{n-1}) \text{ (Claim 19)} \\
&= C_{K_I, K_p, d_2, m}^{(19)} \left(1 + \tau_g^{d_2-m} \right) \\
&\quad \times \left(\left[-y_{n-1}^{\frac{d_2}{d_1}} P \left(\sum_{i=1}^{n-2} Y_i \leq L - y_{n-1} \right) \right]_0^L + \int_0^L P \left(\sum_{i=1}^{n-2} Y_i \leq L - y_{n-1} \right) d \left(y_{n-1}^{\frac{d_2}{d_1}} \right) \right) \\
&= C_{K_I, K_p, d_2, m}^{(19)} \left(1 + \tau_g^{d_2-m} \right) \int_0^L P \left(\sum_{i=1}^{n-2} Y_i \leq L - y_{n-1} \right) \frac{d_2}{d_1} y_{n-1}^{\frac{d_2-d_1}{d_1}} dy_{n-1}.
\end{aligned}$$

By repeating this argument, we get upper bound of $P^{(n)} \left(\sum_{i=1}^{n-1} Y_i \leq L \right)$ as

$$P^{(n)} \left(\sum_{i=1}^{n-1} Y_i \leq L \right) \leq \left(\frac{d_2}{d_1} C_{K_I, K_p, d_2, m}^{(19)} \left(1 + \tau_g^{d_2-m} \right) \right)^{n-1} \int_{\sum_{i=1}^{n-1} y_i \leq L} \prod_{i=1}^{n-1} y_i^{\frac{d_2-d_1}{d_1}} dy.$$

From further upper bounding this, we get upper bound of $P^{(n)} \left(\sum_{i=1}^{n-1} \|X_{i+1} - X_i\|_{\mathbb{R}^m}^{d_1} \leq L \right)$

in (B.5) as

$$\begin{aligned}
& P^{(n)} \left(\sum_{i=1}^{n-1} \|X_{i+1} - X_i\|_{\mathbb{R}^m}^{d_1} \leq L \right) \\
& \leq \left(\frac{d_2}{d_1} C_{K_I, K_p, d_2, m}^{(19)} \left(1 + \tau_g^{d_2-m} \right) \right)^{n-1} \int_{\sum_{i=1}^{n-1} y_i \leq L} \prod_{i=1}^{n-1} y_i^{\frac{d_2-d_1}{d_1}} dy \\
& \leq \left(\frac{2d_2}{d_1} C_{K_I, K_p, d_2, m}^{(19)} \right)^{n-1} L^{\frac{d_2}{d_1}(n-1)} \left(1 + \tau_g^{(d_2-m)(n-1)} \right) \\
& \quad \times \int_{\sum_{i=1}^{n-1} y_i \leq 1} \left(\frac{1}{n-1} \sum_{i=1}^{n-1} y_i \right)^{\frac{(d_2-d_1)(n-1)}{d_1}} dy_{n-1} \cdots dy_1 \\
& = \frac{\left(C_{K_I, K_p, d_1, d_2, m}^{(6)} \right)^{n-1} L^{\frac{d_2}{d_1}(n-1)} \left(1 + \tau_g^{(d_2-m)(n-1)} \right)}{(n-1)^{\left(\frac{d_2}{d_1}-1\right)(n-1)}} \\
& \quad \times \int_0^1 \int_{\sum_{i=1}^{n-2} y_i \leq z} z^{\frac{(d_2-d_1)(n-1)}{d_1}} dy_{n-2} \cdots dy_1 dz \\
& = \frac{\left(C_{K_I, K_p, d_1, d_2, m}^{(6)} \right)^{n-1} L^{\frac{d_2}{d_1}(n-1)} \left(1 + \tau_g^{(d_2-m)(n-1)} \right)}{(n-1)^{\left(\frac{d_2}{d_1}-1\right)(n-1)} (n-2)!} \int_0^1 z^{\frac{d_2(n-1)}{d_1}-1} dz \\
& \leq \frac{\left(C_{K_I, K_p, d_1, d_2, m}^{(6)} \right)^{n-1} L^{\frac{d_2}{d_1}(n-1)} \left(1 + \tau_g^{(d_2-m)(n-1)} \right)}{(n-1)^{\left(\frac{d_2}{d_1}-1\right)(n-1)} (n-1)!},
\end{aligned}$$

where $C_{K_I, K_p, d_1, d_2, m}^{(6)} = \frac{2d_2}{d_1} C_{K_I, K_p, d_2, m}^{(19)}$.

□

Lemma 20. (Space-filling curve) *There exists a surjective map $\psi_d : \mathbb{R} \rightarrow \mathbb{R}^d$ which is Hölder continuous of order $1/d$, i.e.*

$$0 \leq \forall s, t \leq 1, \quad \|\psi_d(s) - \psi_d(t)\|_{\mathbb{R}^d} \leq 2\sqrt{d+3}|s-t|^{1/d}. \quad (\text{B.6})$$

Such a map is called a space-filling curve.

Proof. [See Buchin, 2008, Chapter 2.1.6].

□

Lemma 7. Fix $\tau_g, \tau_\ell \in (0, \infty]$, $K_I \in [1, \infty)$, $K_v \in (0, 2^{-m}]$, $d_1 \in \mathbb{N}$, with $\tau_g \leq \tau_\ell$. Let

$M \in \mathcal{M}_{\tau_g, \tau_\ell, K_p, K_v}^{d_1}$ and $X_1, \dots, X_n \in M$. Then

$$\min_{\sigma \in S_n} \sum_{i=1}^{n-1} \|X_{\sigma(i+1)} - X_{\sigma(i)}\|_{\mathbb{R}^m}^{d_1} \leq C_{K_I, K_v, d_1, m}^{(7)} \left(1 + \tau_g^{d_1-m}\right), \quad (\text{B.7})$$

where $C_{K_I, K_v, d_1, m}^{(7)}$ is a constant depending only on m , d_1 , K_v , and K_I .

Proof. When $d_1 = 1$, length of TSP path is bounded by length of curve $\text{vol}_M(M)$ as in Figure 3.1, and from Lemma 3 we have $\text{vol}_M(M) \leq C_{K_I, d, m}^{(3,2)} (1 + \kappa_g^{m-1})$, hence $C_{K_I, K_v, d_1, m}^{(7)}$ can be set as $C_{K_I, K_v, d_1, m}^{(7)} = C_{K_I, d, m}^{(3,2)}$, as described before.

Consider $d_1 > 1$. By scaling the space-filling curve in Lemma 20, there exists a surjective map $\psi_{d_1} : [0, 1] \rightarrow [-r, r]^{d_1}$ and $\psi_m : [0, 1] \rightarrow [-K_I, K_I]^m$ that satisfies

$$0 \leq \forall s, t \leq 1, \|\psi_{d_1}(s) - \psi_{d_1}(t)\|_{\mathbb{R}^{d_1}} \leq 4r\sqrt{d_1+3}|s-t|^{1/d_1} \quad (\text{B.8})$$

$$0 \leq \forall s, t \leq 1, \|\psi_m(s) - \psi_m(t)\|_{\mathbb{R}^m} \leq 4K_I\sqrt{m+3}|s-t|^{1/m} \quad (\text{B.9})$$

Let $r := 2\sqrt{3}\tau_g$. From Lemma 4, M can be covered by N balls of radius r , denoted by

$$B_M(p_1, r), \dots, B_M(p_N, r), \quad (\text{B.10})$$

with $N = \left\lfloor \frac{2^{d_1} \text{vol}_M(M)}{K_v r^{d_1} \omega_{d_1}} \right\rfloor$. Since $\psi_m : [0, 1] \rightarrow [-K_I, K_I]^m$ in (B.9) is surjective, we can find a right inverse $\Psi_m : [-K_I, K_I]^m \rightarrow [0, 1]$ that satisfies $\psi_m(\Psi_m(p)) = p$, i.e.

$$[0, 1] \xrightleftharpoons[\Psi_m]{\psi_m} [-K_I, K_I]^m. \quad (\text{B.11})$$

Reindex p_k with respect to Ψ_m so that

$$\Psi_m(p_1) < \dots < \Psi_m(p_N). \quad (\text{B.12})$$

Now fix k , and consider the ball $B_M(p_k, r)$ in the covering in (B.10). Then for all $p \in B_M(p_k, r)$, since $d_M(p_k, p) < r$, condition (3) in Definition 2 implies that we can find $\varphi_k(p) \in B_{\mathbb{R}^{d_1}}(0, r)$ such that $\exp_{p_k}(\varphi_k(p)) = p$. So this shows

$$B_M(p_k, r) \subset \exp_{p_k}(B_{\mathbb{R}^{d_1}}(0, r)).$$

Now consider the composition map of the exponential map \exp_{p_k} and ψ_{d_1} in (B.8), $\exp_{p_k} \circ \psi_{d_1} : [0, 1] \rightarrow M$. Then

$$B_M(p_k, r) \subset \exp_{p_k}(B_{\mathbb{R}^{d_1}}(0, r)) \subset \exp_{p_k}([-r, r]^{d_1}) = \exp_{p_k} \circ \psi_{d_1}([0, 1]),$$

where last equality is from that ψ_{d_1} in (B.8) is surjective. So $\exp_{p_k} \circ \psi_{d_1} : [0, 1] \rightarrow M$ is surjective on $B_M(p, r)$, so we can find right inverse $\Psi_k : B_M(p_k, r) \rightarrow [0, 1]$ that satisfies $(\exp_{p_k} \circ \psi_{d_1})(\Psi_k(p)) = p$, i.e.

$$\begin{array}{ccc} [0, 1] & \xrightarrow{\psi_{d_1}} & [-r, r] \xrightarrow{\exp_{p_k}} M \supset B_M(p_k, r). \\ & \xleftarrow{\Psi_k} & \end{array} \quad (\text{B.13})$$

Then, reindex X_1, \dots, X_n with respect to Ψ_m and Ψ_k as $\{X_{k,j}\}_{1 \leq k \leq N, 1 \leq j \leq n_k}$, where $X_{k,1}, \dots, X_{k,n_k} \in B_M(p_k, r)$ and

$$\Psi_k(X_{k,1}) < \dots < \Psi_k(X_{k,n_k}). \quad (\text{B.14})$$

Let $\sigma \in S_n$ be corresponding order of index, so that the d_1 -squared length of the path $\sum_{i=1}^{n-1} \|X_{\sigma(i+1)} - X_{\sigma(i)}\|_{\mathbb{R}^m}^{d_1}$ is factorized as

$$\sum_{i=1}^{n-1} \|X_{\sigma(i+1)} - X_{\sigma(i)}\|_{\mathbb{R}^m}^{d_1} = \sum_{k=1}^N \sum_{j=1}^{n_k-1} \|X_{k,j+1} - X_{k,j}\|_{\mathbb{R}^m}^{d_1} + \sum_{k=1}^{N-1} \|X_{k+1,1} - X_{k,n_k}\|_{\mathbb{R}^m}^{d_1}. \quad (\text{B.15})$$

First, consider the first term $\sum_{k=1}^N \sum_{j=1}^{n_k-1} \|X_{k,j+1} - X_{k,j}\|_{\mathbb{R}^m}^{d_1}$ in (B.15). For all $1 \leq k \leq N$, by applying Lemma 5, $\sum_{j=1}^{n_k-1} \|X_{k,j+1} - X_{k,j}\|_{\mathbb{R}^m}^{d_1}$ is upper bounded as

$$\begin{aligned} & \sum_{j=1}^{n_k-1} \|X_{k,j+1} - X_{k,j}\|_{\mathbb{R}^m}^{d_1} \\ & \leq \sum_{j=1}^{n_k-1} \|(\exp_{p_k} \circ \psi_{d_1})(\Psi_k(X_{k,j+1})) - (\exp_{p_k} \circ \psi_{d_1})(\Psi_k(X_{k,j}))\|_{\mathbb{R}^m}^{d_1} \quad (\text{from (B.13)}) \\ & \leq \left(\frac{\sinh \kappa_l r}{\kappa_l r} \right)^{d_1} \sum_{j=1}^{n_k-1} \|\psi_{d_1}(\Psi_k(X_{k,j+1})) - \psi_{d_1}(\Psi_k(X_{k,j}))\|_{\mathbb{R}^{d_1}}^{d_1} \quad (\text{Lemma 5}) \\ & \leq \left(\frac{4\sqrt{d_1+3} \sinh \kappa_l r}{\kappa_l r} \right)^{d_1} r^{d_1} \sum_{j=1}^{n_k-1} |\Psi_k(X_{k,j+1}) - \Psi_k(X_{k,j})| \quad (\text{from (B.8)}) \\ & \leq \left(\frac{4\sqrt{d_1+3} \sinh \kappa_l r}{\kappa_l r} \right)^{d_1} r^{d_1} \quad (\text{from (B.14)}). \end{aligned}$$

Then, by applying the fact that $\kappa_l r \leq 2\sqrt{3}$ and that $t \mapsto \frac{e^t \sinh t}{t}$ is increasing function on $t \geq 0$ to this, we have upper bound of $\sum_{j=1}^{n_k-1} \|X_{k,j+1} - X_{k,j}\|_{\mathbb{R}^m}^{d_1}$ as

$$\sum_{j=1}^{n_k-1} \|X_{k,j+1} - X_{k,j}\|_{\mathbb{R}^m}^{d_1} \leq \left(\frac{4\sqrt{d_1+3} \sinh 2\sqrt{3}}{2\sqrt{3}} \right)^{d_1} r^{d_1}. \quad (\text{B.16})$$

And then, the second term $\sum_{k=1}^{N-1} \|X_{k+1,1} - X_{k,n_l}\|_{\mathbb{R}^m}^{d_1}$ in (B.15) is upper bounded as

$$\begin{aligned} & \sum_{k=1}^{N-1} \|X_{k+1,1} - X_{k,n_l}\|_{\mathbb{R}^m}^{d_1} \\ & \leq \sum_{k=1}^{N-1} \left(\|X_{k+1,1} - p_{k+1}\|_{\mathbb{R}^m}^{d_1} + \|p_{k+1} - p_k\|_{\mathbb{R}^m}^{d_1} + \|p_k - X_{k,n_l}\|_{\mathbb{R}^m}^{d_1} \right) \\ & \leq 2(N-1)r^{d_1} + \sum_{k=1}^{N-1} \|\psi_m(\Psi_m(p_{k+1})) - \psi_m(\Psi_m(p_k))\|_{\mathbb{R}^{d_1}}^{d_1} \quad (\text{from (B.11)}) \\ & \leq 2(N-1)r^{d_1} + 4\sqrt{m+3}K_I \sum_{k=1}^{N-1} |\Psi_m(p_{k+1}) - \Psi_m(p_k)|^{\frac{d_1}{m}} \quad (\text{from (B.9)}) \\ & \leq 2(N-1)r^{d_1} + 4\sqrt{m+3}K_I \left(\sum_{k=1}^{N-1} |\Psi_m(p_{k+1}) - \Psi_m(p_k)|^{\frac{d_1}{m} \times \frac{m}{d_1}} \right)^{\frac{d_1}{m}} \left(\sum_{k=1}^{N-1} 1^{\frac{m}{m-d_1}} \right)^{\frac{m-d_1}{m}} \\ & \quad (\text{using Hölder's inequality}) \\ & \leq 2(N-1)r^{d_1} + 4\sqrt{m+3}K_I (N-1)^{1-\frac{d_1}{m}} \quad (\text{from (B.12)}). \end{aligned} \quad (\text{B.17})$$

Hence, by plugging in (B.16) and (B.17) to (B.15), we have upper bound on $\sum_{i=1}^{n-1} \|X_{\sigma(i+1)} -$

$X_{\sigma(i)}\|_{\mathbb{R}^m}^{d_1}$ as

$$\begin{aligned}
& \sum_{i=1}^{n-1} \|X_{\sigma(i+1)} - X_{\sigma(i)}\|_{\mathbb{R}^m}^{d_1} \\
& < \left(\left(\frac{4\sqrt{d_1+3} \sinh 2\sqrt{3}}{2\sqrt{3}} \right)^{d_1} + 2 \right) r^{d_1} N + 4\sqrt{m+3} K_I N^{1-\frac{d_1}{m}} \\
& < \frac{2^{d_1} (4\sqrt{d_1+3} \sinh 2\sqrt{3})^{d_1} + 2^{d_1+1}}{K_v \omega_{d_1}} \text{vol}_M(M) \\
& \quad + \frac{4\sqrt{m+3} K_I 2^{d_1(1-\frac{d_1}{m})}}{(K_v (2\sqrt{3})^{d_1} \omega_{d_1})^{1-\frac{d_1}{m}}} \tau_g^{d_1(\frac{d_1}{m}-1)} (\text{vol}_M(M))^{1-\frac{d_1}{m}} \\
& \leq C_{K_I, K_v, d_1, m}^{(7)} \left(1 + \tau_g^{d_1-m} \right),
\end{aligned}$$

by some $C_{K_I, K_v, d_1, m}^{(7)}$ which depends only on m, d_1, K_v , and K_I , where the last line comes from inequality in Lemma 3. Hence we have same upper bound for $\min_{\sigma \in S_n} \sum_{i=1}^{n-1} \|X_{\sigma(i+1)} - X_{\sigma(i)}\|_{\mathbb{R}^m}^{d_1}$ as well, as in (B.7). \square

Proposition 8. Fix $\tau_g, \tau_\ell \in (0, \infty]$, $K_I \in [1, \infty)$, $K_v \in (0, 2^{-m}]$, $K_p \in [(2K_I)^m, \infty)$, $d_1, d_2 \in \mathbb{N}$, with $\tau_g \leq \tau_\ell$ and $1 \leq d_1 < d_2 \leq m$. Then

$$\begin{aligned}
& \inf_{\hat{d}_n} \sup_{P \in \mathcal{P}_1 \cup \mathcal{P}_2} \mathbb{E}_{P^{(n)}} \left[\ell \left(\hat{d}_n, d(P) \right) \right] \\
& \leq \left(C_{K_I, K_p, K_v, d_1, d_2, m}^{(8)} \right)^n \left(1 + \tau_g^{-\left(\frac{d_2}{d_1} m + m - 2d_2\right)n} \right) n^{-\left(\frac{d_2}{d_1}-1\right)n}, \tag{B.18}
\end{aligned}$$

where $C_{K_I, K_p, K_v, d_1, d_2, m}^{(8)} \in (0, \infty)$ is a constant depending only on $K_I, K_p, K_v, d_1, d_2, m$ where

$$\mathcal{P}_1 = \mathcal{P}_{\tau_g, \tau_\ell, K_I, K_v, K_p}^{d_1}, \quad \mathcal{P}_2 = \mathcal{P}_{\tau_g, \tau_\ell, K_I, K_v, K_p}^{d_2}.$$

Proof. Suppose $X = (X_1, \dots, X_n) \in I^n$ is observed, then define $\hat{d}(X)$ as

$$\hat{d}_n(X) := \begin{cases} d_1 & \text{if } \min_{\sigma \in S_n} \left\{ \sum_{i=1}^{n-1} \|X_{\sigma(i+1)} - X_{\sigma(i)}\|_{\mathbb{R}^m}^{d_1} \right\} \leq C_{K_I, K_v, d_1, m}^{(7)} (1 + \tau_g^{d_1-m}) \\ d_2 & \text{otherwise} \end{cases} \tag{B.19}$$

Then for all $P \in \mathcal{P}_{\tau_g, \tau_\ell, K_I, K_v, K_p}^{d_1}$ and $X_1, \dots, X_n \sim P$, by Lemma 7,

$$\min_{\sigma \in S_n} \left\{ \sum_{i=1}^{n-1} \|X_{\sigma(i+1)} - X_{\sigma(i)}\|_{\mathbb{R}^m}^{d_1} \right\} \leq C_{K_I, K_v, d_1, m}^{(7)} \left(1 + \tau_g^{d_1 - m} \right),$$

hence \widehat{d}_n in (B.19) always satisfies $\widehat{d}_n(X) = d_1 = d(P)$, i.e. the risk of \widehat{d}_n satisfies

$$P^{(n)} \left[\widehat{\dim}_n(X_1, \dots, X_n) = d_2 \right] = 0. \quad (\text{B.20})$$

On the other hand, for all $P \in \mathcal{P}_{\tau_g, \tau_\ell, K_I, K_v, K_p}^{d_2}$, the risk of \widehat{d}_n in (B.19) is upper bounded as

$$\begin{aligned} & P^{(n)} \left[\widehat{\dim}_n(X_1, \dots, X_n) = d_1 \right] \\ &= P \left[\bigcup_{\sigma \in S_n} \sum_{i=1}^{n-1} |X_{\sigma(i+1)} - X_{\sigma(i)}| \leq C_{K_I, K_v, d_1, m}^{(7)} \left(1 + \tau_g^{d_1 - m} \right) \right] \\ &\leq \sum_{\sigma \in S_n} P \left[\sum_{i=1}^{n-1} |X_{\sigma(i+1)} - X_{\sigma(i)}| \leq C_{K_I, K_v, d_1, m}^{(7)} \left(1 + \tau_g^{d_1 - m} \right) \right] \\ &= n! P \left[\sum_{i=1}^{n-1} |X_{i+1} - X_i| \leq C_{K_I, K_v, d_1, m}^{(7)} \left(1 + \tau_g^{d_1 - m} \right) \right] \\ &= \frac{n \left(C_{K_p, d_1, d_2, m}^{(2,2)} \right)^{n-1} \left(C_{K_I, K_v, d_1, m}^{(7)} \left(1 + \tau_g^{d_1 - m} \right) \right)^{\frac{d_2}{d_1}(n-1)} \left(1 + \tau_g^{(d_2 - m)(n-1)} \right)}{(n-1) \left(\frac{d_2}{d_1} - 1 \right)^{(n-1)}}, \quad (\text{B.21}) \end{aligned}$$

where last line is implied by Lemma 6. Therefore, by combining (B.20) and (B.21), the minimax rate R_n in (2.6) is upper bounded as in (B.18), as

$$\begin{aligned} & \inf_{\widehat{d}_n} \sup_{P \in \mathcal{P}_{\tau_g, \tau_\ell, K_I, K_v, K_p}^{d_1} \cup \mathcal{P}_{\tau_g, \tau_\ell, K_I, K_v, K_p}^{d_2}} \mathbb{E}_{P^{(n)}} \left[\ell \left(\widehat{d}_n, d(P) \right) \right] \\ &= \frac{n \left(C_{K_p, d_1, d_2, m}^{(2,2)} \left(C_{K_I, K_v, d_1, m}^{(7)} \right)^{\frac{d_2}{d_1}} \right)^{n-1} \left(1 + \tau_g^{-\left(\frac{d_2}{d_1} m + m - 2d_2 \right)(n-1)} \right)}{(n-1) \left(\frac{d_2}{d_1} - 1 \right)^{(n-1)}} \\ &\leq \left(C_{K_I, K_p, K_v, d_1, d_2, m}^{(8)} \right)^n \left(1 + \tau_g^{-\left(\frac{d_2}{d_1} m + m - 2d_2 \right)n} \right) n^{-\left(\frac{d_2}{d_1} - 1 \right)n} \end{aligned}$$

for some $C_{K_I, K_p, K_v, d_1, d_2, m}^{(8)}$ that depends only on $K_I, K_p, K_v, d_1, d_2, m$. \square

C Proofs for Section 4

Lemma 10. Fix $\tau_g, \tau_\ell \in (0, \infty]$, $K_I \in [1, \infty)$, $K_v \in (0, 2^{-m}]$, $d, \Delta d \in \mathbb{N}$, with $\tau_g \leq \tau_\ell$ and $1 \leq d + \Delta d \leq m$. Let $M \in \mathcal{M}_{\tau_g, \tau_\ell, K_I, K_v}^d$ be a d -dimensional manifold of global reach $\geq \tau_g$, local reach $\geq \tau_\ell$, which is embedded in $\mathbb{R}^{m-\Delta d}$. Then

$$M \times [-K_I, K_I]^{\Delta d} \in \mathcal{M}_{\tau_g, \tau_\ell, K_I, K_v}^{d+\Delta d}, \quad (\text{C.1})$$

which is embedded in \mathbb{R}^m .

Proof. For showing (C.1), we need to show 4 conditions in Definition 2. The other conditions are rather obvious and the critical condition is (2), i.e. global reach condition and local reach condition. Showing the local reach condition is almost identical to showing the global reach condition, so we will focus on the global reach condition. From the definition of global reach in Definition 1, we need to show that for all $x \in \mathbb{R}^m$ with $\text{dist}_{\mathbb{R}^m}(x, M \times [-K_I, K_I]^{\Delta d}) < \tau_g$, x has unique closest point $\pi_{M \times [-K_I, K_I]^{\Delta d}}(x)$ on $M \times [-K_I, K_I]^{\Delta d}$.

Let $x \in \mathbb{R}^m$ be satisfying $\text{dist}_{\mathbb{R}^m}(x, M \times [-K_I, K_I]^{\Delta d}) < \tau_g$, and let $y \in M \times [-K_I, K_I]^{\Delta d}$. Then the distance between x and y can be factorized as their distance on first $m - \Delta d$ coordinates and last Δd coordinates,

$$\begin{aligned} \text{dist}_{\mathbb{R}^m}(x, y) \\ = \text{dist}_{\mathbb{R}^{m-\Delta d}}(\Pi_{1:m-\Delta d}(x), \Pi_{1:m-\Delta d}(y)) + \text{dist}_{\mathbb{R}^{\Delta d}}(\Pi_{(m-\Delta d+1):m}(x), \Pi_{(m-\Delta d+1):m}(y)). \end{aligned} \quad (\text{C.2})$$

For the first term in (C.2), note that the projection map $\Pi_{1:m-\Delta d} : \mathbb{R}^m \rightarrow \mathbb{R}^{m-\Delta d}$ is a contradiction, i.e. for all $x, y \in \mathbb{R}^m$, $\text{dist}_{\mathbb{R}^{m-\Delta d}}(\Pi_{1:m-\Delta d}(x), \Pi_{1:m-\Delta d}(y)) \leq \text{dist}_{\mathbb{R}^m}(x, y)$ holds, so $\Pi_{1:m-\Delta d}(x)$ is also within a τ_g -neighborhood of $M = \Pi_{1:m-\Delta d}(M \times [-K_I, K_I]^{\Delta d})$, i.e.

$$\begin{aligned} \text{dist}_{\mathbb{R}^{m-\Delta d}}(\Pi_{1:m-\Delta d}(x), M) &= \text{dist}_{\mathbb{R}^{m-\Delta d}}(\Pi_{1:m-\Delta d}(x), \Pi_{1:m-\Delta d}(M \times [-K_I, K_I]^{\Delta d})) \\ &\leq \text{dist}_{\mathbb{R}^m}(x, M \times [-K_I, K_I]^{\Delta d}) < \tau_g. \end{aligned}$$

Hence from the definition of the global reach in Definition 1, there uniquely exists $\pi_M(\Pi_{1:m-\Delta d}(x)) \in M$. And from $\Pi_{1:m-\Delta d}(y) \in M$, distance of $\Pi_{1:m-\Delta d}(x)$ and $\Pi_{1:m-\Delta d}(y)$ is lower bounded by the distance of $\Pi_{1:m-\Delta d}(x)$ and M , i.e.

$$\begin{aligned} \text{dist}_{\mathbb{R}^{m-\Delta d}}(\Pi_{1:m-\Delta d}(x), \Pi_{1:m-\Delta d}(y)) &\geq \text{dist}_{\mathbb{R}^{m-\Delta d}}(\Pi_{1:m-\Delta d}(x), \pi_M(\Pi_{1:m-\Delta d}(x))) \\ &= \text{dist}_{\mathbb{R}^{m-\Delta d}}(\Pi_{1:m-\Delta d}(x), M), \end{aligned} \quad (\text{C.3})$$

and equality holds if and only if $\Pi_{1:m-\Delta d}(y) = \pi_M(\Pi_{1:m-\Delta d}(x))$.

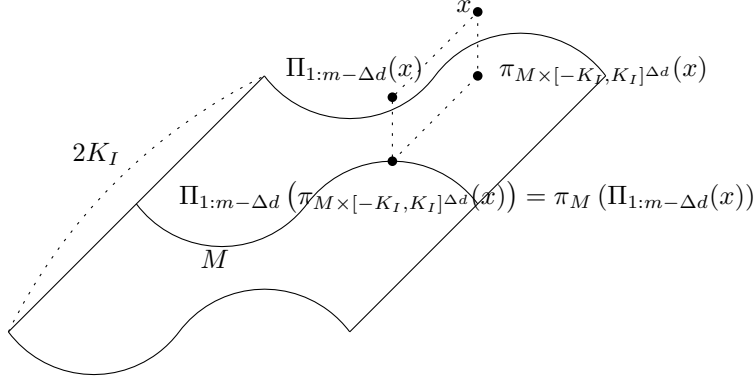


Figure C.1: $\pi_{M \times [-K_I, K_I]^{\Delta d}}(x)$ satisfies $\Pi_{1:m-\Delta d}(\pi_{M \times [-K_I, K_I]^{\Delta d}}(x)) = \pi_M(\Pi_{1:m-\Delta d}(x))$.

The second term in (C.2) is trivially lower bounded by 0, i.e.

$$\text{dist}_{\mathbb{R}^{\Delta d}}(\Pi_{(m-\Delta d+1):m}(x), \Pi_{(m-\Delta d+1):m}(y)) \geq 0, \quad (\text{C.4})$$

and equality holds if and only if $\Pi_{(m-\Delta d+1):m}(x) = \Pi_{(m-\Delta d+1):m}(y)$.

Hence by applying (C.3) and (C.4) to (C.2), $\text{dist}_{\mathbb{R}^m}(x, y)$ is lower bounded by distance of $\Pi_{1:m-\Delta d}(x)$ and M , i.e.

$$\begin{aligned} \text{dist}_{\mathbb{R}^m}(x, y) &= \text{dist}_{\mathbb{R}^{m-\Delta d}}(\Pi_{1:m-\Delta d}(x), \Pi_{1:m-\Delta d}(y)) + \text{dist}_{\mathbb{R}^{\Delta d}}(\Pi_{(m-\Delta d+1):m}(x), \Pi_{(m-\Delta d+1):m}(y)) \\ &\geq \text{dist}_{\mathbb{R}^{m-\Delta d}}(\Pi_{1:m-\Delta d}(x), M), \end{aligned}$$

and equality holds if and only if $\Pi_{1:m-\Delta d}(y) = \pi_M(\Pi_{1:m-\Delta d}(x))$ and $\Pi_{(m-\Delta d+1):m}(x) = \Pi_{(m-\Delta d+1):m}(y)$, i.e. when $y = (\pi_M(\Pi_{1:m-\Delta d}(x)), \Pi_{(m-\Delta d+1):m}(y))$. Hence x has unique closest point $\pi_{M \times [-K_I, K_I]^{\Delta d}}(x)$ on $M \times [-K_I, K_I]^{\Delta d}$ as

$$\pi_{M \times [-K_I, K_I]^{\Delta d}}(x) = (\pi_M(\Pi_{1:m-\Delta d}(x)), \Pi_{(m-\Delta d+1):m}(x)),$$

as in Figure C.1. □

Lemma 11. Fix $\tau_\ell \in (0, \infty]$, $K_I \in [1, \infty)$, $d_1, d_2 \in \mathbb{N}$, with $1 \leq d_1 \leq d_2$, and suppose $2\tau_\ell \leq K_I$. Then there exist $T_1, \dots, T_n \subset [-K_I, K_I]^{d_2}$ such that:

- (1) The T_i 's are distinct.
- (2) For each T_i , there exists an isometry Φ_i such that

$$T_i = \Phi_i\left([-K_I, K_I]^{d_1-1} \times [0, a] \times B_{\mathbb{R}^{d_2-d_1}}(0, w)\right), \quad (\text{C.5})$$

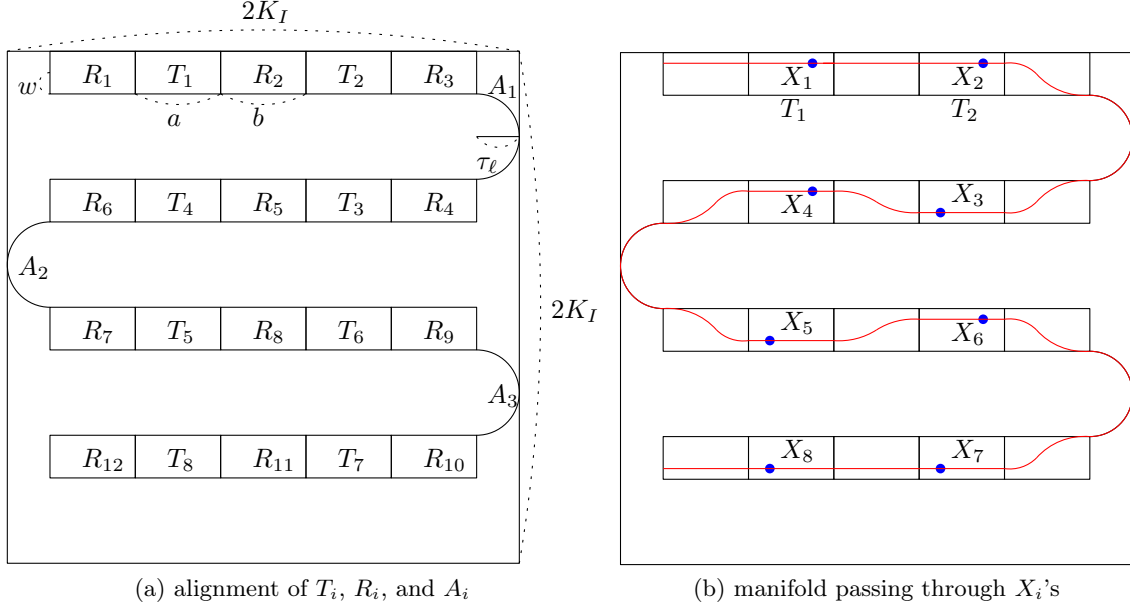


Figure C.2: This figure illustrates the case where $d_1 = 1$ and $d_2 = 2$. (a) shows how T_i , R_i , and A_i 's are aligned in a zigzag. (b) shows for given $X_1 \in T_1, \dots, X_n \in T_n$ (represented as blue points), how $\mathcal{M}(\{\Pi_{(d_1+1):d_2}^{-1} \Phi_i^{-1}(X_i)\}_{1 \leq i \leq n})$ (represented as a red curve) passes through X_1, \dots, X_n .

where $c = \left\lceil \frac{K_I + \tau_\ell}{2\tau_\ell} \right\rceil$, $a = \frac{K_I - \tau_\ell}{(d + \frac{1}{2}) \left\lceil \frac{n}{c^{d_2 - d_1}} \right\rceil}$, and $w = \min \left\{ \tau_\ell, \frac{d^2 (K_I - \tau_\ell)^2}{2\tau_\ell (d + \frac{1}{2})^2 \left(\left\lceil \frac{n}{c^{d_2 - d_1}} \right\rceil + 1 \right)^2} \right\}$.

(3) There exists $\mathcal{M} : (B_{\mathbb{R}^{d_2 - d_1}}(0, w))^n \rightarrow \mathcal{M}_{\tau_g, \tau_\ell, K_I, K_v}^{d_1}$ one-to-one such that for each $Y_i \in B_{\mathbb{R}^{d_2 - d_1}}(0, w)$, $1 \leq i \leq n$, $\mathcal{M}(Y_1, \dots, Y_n) \cap T_i = \Phi_i([-K_I, K_I]^{d_1 - 1} \times [0, a] \times \{Y_i\})$. Hence for any $X_1 \in T_1, \dots, X_n \in T_n$, $\mathcal{M}(\{\Pi_{(d_1+1):d_2}^{-1} \Phi_i^{-1}(X_i)\}_{1 \leq i \leq n})$ passes through X_1, \dots, X_n .

Proof. By Lemma 10, we only need to show the case for $d_1 = 1$. Let $b = \frac{2d(K_I - \tau_\ell)}{(d + \frac{1}{2}) \left(\left\lceil \frac{n}{c^{d_2 - d_1}} \right\rceil + 1 \right)}$, so that $b \geq 2\sqrt{2w\tau_\ell}$ and $2\tau_\ell + \left\lfloor \frac{n}{c^{d_2 - d_1}} \right\rfloor a + \left(\left\lfloor \frac{n}{c^{d_2 - d_1}} \right\rfloor + 1 \right) b = 2K_I$ holds. With such values of a , b , and w , align T_i , R_i , and A_i in a zigzag way, as in Figure C.2(a).

Then from the definition of T_i , it is apparent that (1) the T_i 's are distinct and (2) for each T_i , there exists an isometry Φ_i such that $T_i = \Phi_i([-K_I, K_I]^{d_1 - 1} \times [0, a] \times B_{\mathbb{R}^{d_2 - d_1}}(0, w))$. There exists isometry Ψ_i such that $R_i = \Psi_i([-K_I, K_I]^{d_1 - 1} \times [0, b] \times B_{\mathbb{R}^{d_2 - d_1}}(0, w))$ as well. Hence condition (1) and (2) are satisfied.

We are left to define \mathcal{M} that satisfies condition (3). Now define $\mathcal{M} : (B_{\mathbb{R}^{d_2 - d_1}}(0, w))^n \rightarrow$

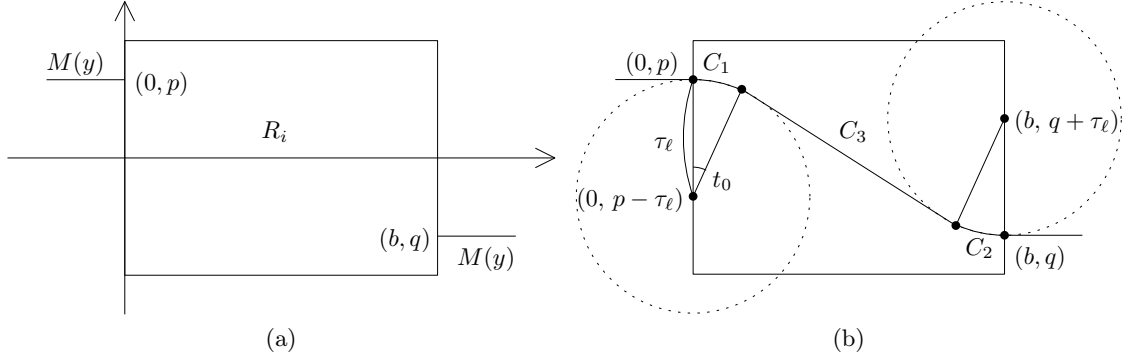


Figure C.3: (a) We need to find C^2 curve with local reach $\geq \tau_\ell$ that starts from $(0, p) \in \mathbb{R}^2$, ends at (b, q) , and velocity at each end points are both parallel to $(1, 0)$. (b) C_1 and C_2 are arcs of circles of radius R_l , and C_3 is the cotangent segment of two circles.

$\mathcal{M}_{\tau_g, \tau_\ell, K_I, K_v}^{d_1}$ as follows. For each $Y_i \in B_{\mathbb{R}^{d_2-d_1}}(0, w)$, $1 \leq i \leq n$, $\bigcup_{i=1}^4 A_i \subset \mathcal{M}(Y_1, \dots, Y_n) \subset \left(\bigcup_{i=1}^4 A_i \right) \cup \left(\bigcup_{i=1}^4 T_i \right) \cup \left(\bigcup_{i=1}^4 R_i \right)$. The intersection of $\mathcal{M}(Y_1, \dots, Y_n)$ and T_i is a line segment $\Phi_i([-K_I, K_I]^{d_1-1} \times [0, a] \times \{Y_i\})$. Our goal is to make $\mathcal{M}(Y_1, \dots, Y_n)$ be C^1 and piecewise C^2 .

See Figure C.3 for construction of intersection of $\mathcal{M}(Y_1, \dots, Y_n)$ and R_i . Given that $\mathcal{M}(Y_1, \dots, Y_n) \cap \left(\left(\bigcup_{i=1}^4 A_i \right) \cup \left(\bigcup_{i=1}^4 T_i \right) \right)$ is determined, two points on $\mathcal{M}(Y_1, \dots, Y_n) \cap \partial R_i$ is already determined. By translation and rotation if necessary, for all p, q with $-w \leq q \leq p \leq w$, we need to find C^2 curve with curvature $\leq \tau_\ell$ that starts from $(0, p) \in \mathbb{R}^2$, ends at $(b, q) \in \mathbb{R}^2$, and velocity at each end points are both parallel to $(1, 0) \in \mathbb{R}^2$, as in Figure C.3(a).

Let

$$t_0 = \cos^{-1} \left(\frac{2\tau_\ell(2\tau_\ell - (p - q)) + b\sqrt{b^2 - (p - q)(4\tau_\ell - (p - q))}}{b^2 + (2\tau_\ell - (p - q))^2} \right), \quad (\text{C.6})$$

and let

$$C_1 = \{(0, p - \tau_\ell) + \tau_\ell(\sin t, \cos t) \mid 0 \leq t \leq t_0\}.$$

Then C_1 is an arc of circle of which center is $(0, p - \tau_\ell)$, and starts at $(0, p)$ when $t = 0$ and ends at $(\tau_\ell \sin t_0, p - \tau_\ell(1 - \cos t_0))$ when $t = t_0$. Also, the normalized velocities of C_1 at endpoints are

$$(1, 0) \text{ at } (0, p), \quad (\cos t_0, -\sin t_0) \text{ at } (\tau_\ell \sin t_0, p - \tau_\ell(1 - \cos t_0)). \quad (\text{C.7})$$

Similarly, let

$$C_2 = \{(b, q + \tau_\ell) - \tau_\ell(\sin t, \cos t) \mid 0 \leq t \leq t_0\}.$$

Then C_2 is an arc of a circle of whose center is $(b, q + \tau_\ell)$, and starts at (b, q) when $t = 0$ and ends at $(b - \tau_\ell \sin t_0, q + \tau_\ell(1 - \cos t_0))$ when $t = t_0$. Also, the normalized velocities of C_2 at endpoints are

$$(-1, 0) \text{ at } (b, q), \quad (-\cos t_0, \sin t_0) \text{ at } (b - \tau_\ell \sin t_0, q + \tau_\ell(1 - \cos t_0)). \quad (\text{C.8})$$

Let

$$C_3 = \{(1 - s)(\tau_\ell \sin t_0, p - \tau_\ell(1 - \cos t_0)) + s(b - \tau_\ell \sin t_0, q + \tau_\ell(1 - \cos t_0)) \mid 0 \leq s \leq 1\},$$

so that C_3 is a segment joining $(\tau_\ell \sin t_0, p - \tau_\ell(1 - \cos t_0))$ (when $s = 0$) and $(b - \tau_\ell \sin t_0, q + \tau_\ell(1 - \cos t_0))$ (when $s = 1$). Also, its velocity vector is

$$(b - \tau_\ell \sin t_0, q + \tau_\ell(1 - \cos t_0)) \text{ for all } s \in [0, 1]. \quad (\text{C.9})$$

Then from definition of t_0 in (C.6),

$$\cos t_0 (q - p + 2\tau_\ell(1 - \cos t_0)) + \sin t_0 (b - 2\tau_\ell \sin t_0) = 0,$$

and this implies that $(b - 2\tau_\ell \sin t_0, q - p + 2\tau_\ell(1 - \cos t_0))$ is parallel to $(\cos t_0, -\sin t_0)$. Hence the velocity vector of C_3 in (C.9) is parallel to the velocity vector of C_1 in (C.7) at $(\tau_\ell \sin t_0, p - \tau_\ell(1 - \cos t_0))$ and the velocity vector of C_2 in (C.8) at $(b - \tau_\ell \sin t_0, q + \tau_\ell(1 - \cos t_0))$, i.e. C_3 is cotangent to both C_1 and C_2 . See Figure C.3(b).

Now we check whether $\mathcal{M}(Y_1, \dots, Y_n)$ is of global reach $\geq \tau_g$. Now from the construction, a ball of radius τ_g always circumscribes $\mathcal{M}(Y_1, \dots, Y_n)$ at any point. Refer to Figure C.3(b). Hence, for all $x \in \mathcal{M}(Y_1, \dots, Y_n)$ and for any $y \in \mathbb{R}^m$ such that $y - x \perp T_x \mathcal{M}(Y_1, \dots, Y_n)$ and $\|y - x\|_2 = \tau_g$, $B_{\mathbb{R}^m}(y, \tau_g) \cap \mathcal{M}(Y_1, \dots, Y_n) = \emptyset$. Hence from Lemma 2, $\mathcal{M}(Y_1, \dots, Y_n)$ is of global reach $\geq \tau_g$. \square

Claim 12. Let $T = S_n \prod_{i=1}^n T_i$ where the T_i 's are from Lemma 11, and let Q_1, Q_2 be from (C.15) in Proposition 13. Then for all $x \in \text{int}T$, there exists $r_x > 0$ such that for all $r < r_x$,

$$Q_1 \left(\prod_{i=1}^n B_{\|\cdot\|_{\mathbb{R}^{d_2}, \infty}}(x_i, r) \right) \geq \frac{\tau_\ell^{(d_2 - d_1)n}}{2^n K_I^{(d_2 - d_1)n}} Q_2 \left(\prod_{i=1}^n B_{\|\cdot\|_{\mathbb{R}^{d_2}, \infty}}(x_i, r) \right). \quad (\text{C.10})$$

Proof. By symmetry, we can assume that $x \in \prod_{i=1}^n T_i$, i.e. $x_1 \in T_1, \dots, x_n \in T_n$. Choose r_x small enough so that $B(x, r_x) \subset \text{int}U_n$. Then for all $r < r_x$, from the definition of Q_1 in

(C.15),

$$\begin{aligned}
Q_1 \left(\prod_{i=1}^n B_{\|\cdot\|_{\mathbb{R}^{d_2}, \infty}}(x_i, r) \right) &= \int_{\mathcal{P}_1} P^{(n)} \left(\prod_{i=1}^n B_{\|\cdot\|_{\mathbb{R}^{d_2}, \infty}}(x_i, r) \right) d\mu_1(P) \\
&= \int_{C^n} \Phi(y)^{(n)} \left(\prod_{i=1}^n B_{\|\cdot\|_{\mathbb{R}^{d_2}, \infty}}(x_i, r) \right) \lambda_{C^n}(y) \\
&= \int_{C^n} \prod_{i=1}^n \lambda_{\mathcal{M}(y)} \left(B_{\|\cdot\|_{\mathbb{R}^{d_2}, \infty}}(x_i, r) \right) \lambda_{C^n}(y). \tag{C.11}
\end{aligned}$$

Then from condition (3) in Lemma 11, $\mathcal{M}(y) \cap T_i = \Phi_i([-K_I, K_I]^{d_1-1} \times [0, a] \times \{y_i\})$ holds, hence

$$\begin{aligned}
&\mathcal{M}(y) \cap B_{\|\cdot\|_{\mathbb{R}^{d_2}, \infty}}(x_i, r) \\
&= \begin{cases} \Phi_i \left(B_{\|\cdot\|_{\mathbb{R}^{d_2}, \infty}}(\Pi_{1:d_1}(\Phi_i^{-1}(x_i)), r) \times \{y_i\} \right) & \text{if } \|y_i - \Pi_{(d_1+1):d_2}(\Phi_i^{-1}(x_i))\|_{\mathbb{R}^{d_2-d_1}} < r \\ \emptyset & \text{otherwise.} \end{cases}
\end{aligned}$$

And hence the volume of $\mathcal{M}(y) \cap B_{\|\cdot\|_{\mathbb{R}^{d_2}, \infty}}(x_i, r)$ can be lower bounded as

$$\lambda_{\mathcal{M}(y)} \left(B_{\|\cdot\|_{\mathbb{R}^{d_2}, \infty}}(x_i, r) \right) \geq \frac{r^{d_1}}{2K_I^{d_1-1}an} I \left(\|y_i - \Pi_{(d_1+1):d_2}(\Phi_i^{-1}(x_i))\|_{\mathbb{R}^{d_2-d_1}, \infty} < r \right).$$

By applying this to (C.11), $Q_1 \left(\prod_{i=1}^n B_{\|\cdot\|_{\mathbb{R}^{d_2}, \infty}}(x_i, r) \right)$ can be lower bounded as

$$\begin{aligned}
&Q_1 \left(\prod_{i=1}^n B_{\|\cdot\|_{\mathbb{R}^{d_2}, \infty}}(x_i, r) \right) \\
&\geq \int_{C^n} \prod_{i=1}^n \frac{r^{d_1}}{2K_I^{d_1-1}an} I \left(\|y_i - \Pi_{(d_1+1):d_2}(\Phi_i^{-1}(x_i))\|_{\mathbb{R}^{d_2-d_1}, \infty} < r \right) \lambda_{C^n}(y) \\
&= \frac{r^{d_1n}}{2^n K_I^{(d_1-1)n} (an)^n} \prod_{i=1}^n \int_C I \left(\|y_i - \Pi_{(d_1+1):d_2}(\Phi_i^{-1}(x_i))\|_{\mathbb{R}^{d_2-d_1}, \infty} < r \right) \lambda_C(y_i) \\
&= \frac{r^{d_1n}}{2^n K_I^{(d_1-1)n} (an)^n} \left(\frac{(2r)^{d_2-d_1}}{w^{d_2-d_1} \omega_{d_2-d_1}} \right)^n \\
&= \frac{2^{(d_2-d_1-1)n} r^{d_2n}}{K_I^{(d_1-1)n} w^{(d_2-d_1)n} (an)^n \omega_{d_2-d_1}^n} \\
&\geq \frac{2^{(d_2-d_1-1)n} \tau_\ell^{(d_2-d_1)n} r^{d_2n}}{K_I^{(2d_2-d_1)n} \omega_{d_2-d_1}^n}, \tag{C.12}
\end{aligned}$$

where the last inequality uses $an \leq c^d K_I \leq \frac{K_I^{d_2-d_1+1}}{\tau_\ell^{d_2-d_1}}$ and $w \leq K_I$.

On the other hand, $Q_2 \left(\prod_{i=1}^n B_{\|\cdot\|_{\mathbb{R}^{d_2}, \infty}}(x_i, r) \right) = \left(\frac{2r}{2K_I} \right)^{d_2 n} = \frac{r^{d_2 n}}{K_I^{d_2 n}}$, so from this and (C.12), we get (C.10) as

$$\begin{aligned} Q_1 \left(\prod_{i=1}^n B_{\|\cdot\|_{\mathbb{R}^{d_2}, \infty}}(x_i, r) \right) &\geq \frac{2^{(d_2-d_1-1)n} \tau_\ell^{(d_2-d_1)n}}{K_I^{(d_2-d_1)n} \omega_{d_2-d_1}^n} Q_2 \left(\prod_{i=1}^n B_{\|\cdot\|_{\mathbb{R}^{d_2}, \infty}}(x_i, r) \right) \\ &\geq \frac{\tau_\ell^{(d_2-d_1)n}}{2^n K_I^{(d_2-d_1)n}} Q_2 \left(\prod_{i=1}^n B_{\|\cdot\|_{\mathbb{R}^{d_2}, \infty}}(x_i, r) \right). \end{aligned}$$

□

Proposition 13. Fix $\tau_g, \tau_\ell \in (0, \infty]$, $K_I \in [1, \infty)$, $K_v \in (0, 2^{-m}]$, $K_p \in [(2K_I)^m, \infty)$, $d_1, d_2 \in \mathbb{N}$, with $\tau_g \leq \tau_\ell$ and $1 \leq d_1 < d_2 \leq m$, and suppose that $2\tau_\ell < K_I$. Then

$$\begin{aligned} &\inf_{\widehat{d}} \sup_{P \in \mathcal{Q}} \mathbb{E}_{P^{(n)}}[\ell(\widehat{d}_n, d(P))] \\ &\geq \left(C_{d_1, d_2, K_I}^{(13)} \right)^n \tau_\ell^{(d_2-d_1)n} \min \left\{ \tau_\ell^{-2(d_2-d_1)-1} n^{-2}, 1 \right\}^{(d_2-d_1)n}, \end{aligned} \quad (\text{C.13})$$

where $C_{d_1, d_2, K_I}^{(13)} \in (0, \infty)$ is a constant depending only on d_1, d_2 , and K_I and

$$\mathcal{Q} = \mathcal{P}_{\tau_g, \tau_\ell, K_I, K_v, K_p}^{d_1} \bigcup \mathcal{P}_{\tau_g, \tau_\ell, K_I, K_v, K_p}^{d_2}.$$

Proof. Let $J = [-K_I, K_I]^{d_2}$. Let S_n be the permutation group, and $S_n \curvearrowright J^n$ by coordinate change, i.e. $\sigma \in S_n$, $x \in J^n$, $\sigma x := (x_{\sigma(1)}, \dots, x_{\sigma(n)})$. For any set $A \subset J^n$, let $S_n A := \{\sigma x \in J^n : \sigma \in S_n, x \in A\}$.

Let T_i be T_i 's from Lemma 11. Let $T := S_n \prod_{i=1}^n T_i$, and $V := \bigcup_{i=1}^n T_i = \Pi_{1:d_2}(T)$. Intuitively, T is the set of points $x = (x_1, \dots, x_n)$ where x_i lies on one of the T_j .

Let $C = B_{\mathbb{R}^{d_2-d_1}}(0, w)$ where w is from Lemma 11, and precisely define a set of d_1 -dimensional distribution \mathcal{P}_1 in (4.1) and a set of d_2 -dimensional distribution \mathcal{P}_2 in (4.2) as

$$\begin{aligned} \mathcal{P}_1 &= \{P \in \mathcal{P}_{\tau_g, \tau_\ell, K_I, K_v, K_p}^{d_1} : \text{there exists } M \in \mathcal{M}(C^n) \text{ such that } P \text{ is uniform on } M\}, \\ \mathcal{P}_2 &= \{\lambda_J\} \subset \mathcal{P}_{\tau_g, \tau_\ell, K_I, K_v, K_p}^{d_2}. \end{aligned} \quad (\text{C.14})$$

Define a map $\Phi : C^n \rightarrow \mathcal{P}_1$ by $\Phi(y_1, \dots, y_n) = \lambda_{\mathcal{M}(y_1, \dots, y_n)}$, i.e. the uniform measure on $\mathcal{M}(y_1, \dots, y_n)$. Impose a topology and probability measure structure on \mathcal{P}_1 by the pushforward topology and the uniform measure on C^n , i.e. $\mathcal{P}' \subset \mathcal{P}_1$ is open if and only if $\Phi^{-1}(\mathcal{P}')$ is open in C^n , $\mathcal{P}' \subset \mathcal{P}_1$ is measurable if and only if $\Phi^{-1}(\mathcal{P}') \in \mathcal{B}(C^n)$, and $\mu_1(\mathcal{P}') = \lambda_{C^n}(\Phi^{-1}(\mathcal{P}'))$.

Define a probability measure Q_1, Q_2 on $(J^n, \mathcal{B}(J^n))$ by

$$Q_1(A) := \int_{\mathcal{P}_1} P^{(n)}(A) d\mu_1(P) \quad \text{and} \quad Q_2 = \lambda_{J^n}. \quad (\text{C.15})$$

Fix $P \in \mathcal{P}_1$, let $x = \Phi^{-1}(P)$. Then $P^{(n)}(A) = \lambda_{\mathcal{M}(x)}^{(n)}(A)$ is a measurable function of x and Φ is a homeomorphism. Hence, $p^{(n)}(A)$ is measurable function and $Q_1(A)$ is well defined. Define $\nu = Q_1 + \lambda_J$. Then $Q_1, Q_2 \ll \nu$, so there exist densities $q_1 = \frac{dQ_1}{d\nu}$, $q_2 = \frac{dQ_2}{d\nu}$ with respect to ν .

Then by applying Le Cam's Lemma (Lemma 9) with $\theta(P) = d(P)$, \mathcal{P}_1 and \mathcal{P}_2 from (C.14), and Q_1 and Q_2 in (C.15), the minimax rate $\inf_{\hat{d}} \sup_{P \in \mathcal{P}_1 \cup \mathcal{P}_2} \mathbb{E}_P [\ell(\hat{d}_n, d(P))]$ can be lower bounded as

$$\begin{aligned} \inf_{\hat{d}} \sup_{P \in \mathcal{P}_1 \cup \mathcal{P}_2} \mathbb{E}_P [\ell(\hat{d}_n, d(P))] &\geq \frac{\ell(d_1, d_2)}{4} \int_{J^n} q_1(x) \wedge q_2(x) d\nu(x) \\ &= \frac{1}{4} \int_{J^n} q_1(x) \wedge q_2(x) d\nu(x). \end{aligned} \quad (\text{C.16})$$

Then from Claim 12, for all $x \in \text{int}T$, there exists $r_x > 0$ s.t. for all $r < r_x$,

$$Q_1 \left(\prod_{i=1}^n B_{\|\cdot\|_{\mathbb{R}^{d_2}, \infty}}(x_i, r) \right) \geq \frac{\tau_\ell^{(d_2-d_1)n}}{2^n K_I^{(d_2-d_1)n}} Q_2 \left(\prod_{i=1}^n B_{\|\cdot\|_{\mathbb{R}^{d_2}, \infty}}(x_i, r) \right).$$

Hence $q_1(x)$ is lower bounded by $q_2(x)$ whenever $x \in T$ as

$$q_1(x) \geq \frac{\tau_\ell^{(d_2-d_1)n}}{2^n K_I^{(d_2-d_1)n}} q_2(x) \text{ if } x \in T,$$

and since $2\tau_\ell \leq K_I$, $q_1(x) \wedge q_2(x)$ is correspondingly lower bounded by $q_2(x)$ as

$$q_1(x) \wedge q_2(x) \geq \frac{\tau_\ell^{(d_2-d_1)n}}{2^n K_I^{(d_2-d_1)n}} q_2(x) 1(x \in T).$$

Hence the integration of $q_1(x) \wedge q_2(x)$ over T is lower bounded as

$$\frac{1}{4} \int_T q_1(x) \wedge q_2(x) d\nu(x) \geq \frac{\tau_\ell^{(d_2-d_1)n}}{2^n K_I^{(d_2-d_1)n}} \lambda_{J^n}(T). \quad (\text{C.17})$$

Then from $a = \frac{K_I - \tau_\ell}{(d + \frac{1}{2}) \lceil \frac{n}{c^{d_2 - d_1}} \rceil}$ and $w = \min \left\{ \tau_\ell, \frac{d^2(K_I - \tau_\ell)^2}{2\tau_\ell(d + \frac{1}{2})^2 \left(\lceil \frac{n}{c^{d_2 - d_1}} \rceil + 1 \right)^2} \right\}$, $\lambda_{J^n}(T)$ can be lower bounded as

$$\begin{aligned} \lambda_{J^n} \left(S_n \prod_{i=1}^n T_i \right) &= n! \lambda_{J^1}(T_1)^n \\ &= n! \left(\frac{(2K_I)^{d_1 - 1} \omega_{d_2 - d_1} a w^{d_2 - d_1}}{(2K_I)^{d_2}} \right)^n \\ &\geq \left(C_{d_1, d_2, K_I}^{(13,1)} \right)^n \min \left\{ \tau_\ell^{-2(d_2 - d_1) - 1} n^{-2}, 1 \right\}^{(d_2 - d_1)n}, \end{aligned} \quad (\text{C.18})$$

for some constant $C_{d_1, d_2, K_I}^{(13,1)}$ that depends only on d_1 , d_2 , and K_I . Hence by combining (C.16), (C.17), and (C.18), the minimax rate $\inf_{\hat{d}} \sup_{P \in \mathcal{P}_1 \cup \mathcal{P}_2} \mathbb{E}_P \left[\ell(\hat{d}_n, d(P)) \right]$ can be lower bounded as

$$\inf_{\hat{d}} \sup_{P \in \mathcal{P}_1 \cup \mathcal{P}_2} \mathbb{E}_P \left[\ell(\hat{d}_n, d(P)) \right] \geq \left(C_{d_1, d_2, K_I}^{(13)} \right)^n \tau_\ell^{(d_2 - d_1)n} \min \left\{ \tau_\ell^{-2(d_2 - d_1) - 1} n^{-2}, 1 \right\}^{(d_2 - d_1)n},$$

for some constant $C_{d_1, d_2, K_I}^{(13)}$ that depends only on d_1 , d_2 , and K_I . Then since $\mathcal{P}_1 \subset \mathcal{P}_{\tau_g, \tau_\ell, K_I, K_v, K_p}^{d_1}$ and $\mathcal{P}_2 \subset \mathcal{P}_{\tau_g, \tau_\ell, K_I, K_v, K_p}^{d_2}$, the minimax rate R_n in (2.6) can be lower bounded by the minimax rate $\inf_{\hat{d}} \sup_{P \in \mathcal{P}_1 \cup \mathcal{P}_2} \mathbb{E}_P \left[\ell(\hat{d}_n, d(P)) \right]$, i.e.

$$\inf_{\substack{\hat{d}_n \\ \dim P \in \mathcal{P}_{\tau_g, \tau_\ell, K_I, K_v, K_p}^{d_1} \cup \mathcal{P}_{\tau_g, \tau_\ell, K_I, K_v, K_p}^{d_2}}} \sup \mathbb{E}_P [\ell(\widehat{\dim}_n, \dim(P))] \geq \inf_{\dim P \in \mathcal{P}_1 \cup \mathcal{P}_2} \sup \mathbb{E}_P [\ell(\widehat{\dim}_n, \dim(P))],$$

which completes the proof of showing (C.13). \square

D Proofs For Section 5

Proposition 14. Fix $\tau_g, \tau_\ell \in (0, \infty]$, $K_I \in [1, \infty)$, $K_v \in (0, 2^{-m}]$, $K_p \in [(2K_I)^m, \infty)$, $d_1, d_2 \in \mathbb{N}$, with $\tau_g \leq \tau_\ell$ and $1 \leq d_1 < d_2 \leq m$. Then:

$$\inf_{\hat{d}_n} \sup_{P \in \mathcal{P}} \mathbb{E}_{P^{(n)}} \left[\ell(\hat{d}_n, d(P)) \right] \leq \left(C_{K_I, K_p, K_v, m}^{(14)} \right)^n \left(1 + \tau_g^{-(m^2 - m)n} \right) n^{-\frac{1}{m-1}n} \quad (\text{D.1})$$

where $C_{K_I, K_p, K_v, m}^{(14)} \in (0, \infty)$ is a constant depending only on K_I, K_p, K_v, m .

Proof. Suppose $X = (X_1, \dots, X_n) \in I^n$ is observed, then define $\widehat{d}_n(X)$ as

$$\widehat{d}_n(X) := \min \left\{ d \in [1, m] : \min_{\sigma \in S_n} \left\{ \sum_{i=1}^{n-1} \|X_{\sigma(i+1)} - X_{\sigma(i)}\|_{\mathbb{R}^m}^d \right\} \leq C_{K_I, K_v, d, m}^{(7)} \left(1 + \tau_g^{d-m} \right) \right\} \quad (\text{D.2})$$

Then for all $P \in \mathcal{P}_{\tau_g, \tau_\ell, K_I, K_v, K_p}^d$ and $X_1, \dots, X_n \sim P$, by Lemma 7,

$$\min_{\sigma \in S_n} \left\{ \sum_{i=1}^{n-1} \|X_{\sigma(i+1)} - X_{\sigma(i)}\|_{\mathbb{R}^m}^d \right\} \leq C_{K_I, K_v, d, m}^{(7)} \left(1 + \tau_g^{d-m} \right),$$

hence \widehat{d}_n in (D.2) always satisfies

$$\widehat{d}_n(X) \leq d = d(P). \quad (\text{D.3})$$

From (D.3) and Proposition 8, the risk of \widehat{d}_n in (D.2) is upper bounded as

$$\begin{aligned} & P^{(n)} \left[\widehat{\dim}_n(X_1, \dots, X_n) \neq d \right] \\ &= P^{(n)} \left[\max \left\{ k \in [1, m] : \min_{\sigma \in S_n} \left\{ \sum_{i=1}^{n-1} \|X_{\sigma(i+1)} - X_{\sigma(i)}\|_{\mathbb{R}^m}^k \right\} \leq C_{K_I, K_v, d, m}^{(7)} \left(1 + \tau_g^{k-m} \right) \right\} \right. \\ & \quad \left. < d \right] \text{ (from (D.3))} \\ &\leq \sum_{k=1}^{d-1} P^{(n)} \left[\min_{\sigma \in S_n} \left\{ \sum_{i=1}^{n-1} \|X_{\sigma(i+1)} - X_{\sigma(i)}\|_{\mathbb{R}^m}^k \right\} \leq C_{K_I, K_v, k, m}^{(7)} \left(1 + \tau_g^{k-m} \right) \right] \\ &\leq \sum_{k=1}^{d-1} \left(C_{K_I, K_p, K_v, k, d, m}^{(8)} \right)^n \left(1 + \tau_g^{-\left(\frac{d}{k}m + m - 2d\right)n} \right) n^{-\left(\frac{d}{k}-1\right)n} \text{ (Proposition 8)} \\ &\leq \left(C_{K_I, K_p, K_v, m}^{(14)} \right)^n \left(1 + \tau_g^{-(m^2-m)n} \right) n^{-\frac{1}{m-1}n}, \end{aligned}$$

for some $C_{K_I, K_p, K_v, m}^{(14)}$ that depends only on K_I, K_p, K_v, m . Therefore, the minimax rate R_n in (2.6) is upper bounded as in (D.1), as

$$\inf_{\widehat{d}_n} \sup_{P \in \mathcal{P}} \mathbb{E}_{P^{(n)}} \left[\ell \left(\widehat{d}_n, d(P) \right) \right] \leq \left(C_{K_I, K_p, K_v, m}^{(14)} \right)^n \left(1 + \tau_g^{-(m^2-m)n} \right) n^{-\frac{1}{m-1}n}. \quad (\text{D.4})$$

□

Proposition 15. Fix $\tau_g, \tau_\ell \in (0, \infty]$, $K_I \in [1, \infty)$, $K_v \in (0, 2^{-m}]$, $K_p \in [(2K_I)^m, \infty)$, $d_1, d_2 \in \mathbb{N}$, with $\tau_g \leq \tau_\ell$ and $1 \leq d_1 < d_2 \leq m$, and suppose that $\tau_\ell < K_I$. Then,

$$\inf_{\widehat{d}} \sup_{P \in \mathcal{P}} \mathbb{E}_{P^{(n)}}[\ell(\widehat{d}_n, d(P))] \geq \left(C_{K_I}^{(15)}\right)^n \tau_\ell^n \min\{\tau_\ell^{-3} n^{-2}, 1\}^n \quad (\text{D.5})$$

where $C_{K_I}^{(15)} \in (0, \infty)$ is a constant depending only on K_I .

Proof. For any d_1 and d_2 , from Proposition 13,

$$\begin{aligned} & \inf_{\widehat{\dim} P \in \mathcal{P}} \sup_{P^{(n)}} [\ell(\widehat{\dim}_n, \dim(P))] \\ & \geq \inf_{\widehat{\dim} P \in \mathcal{P}_{\tau_g, \tau_\ell, K_I, K_v, K_p}^{d_1} \cup \mathcal{P}_{\tau_g, \tau_\ell, K_I, K_v, K_p}^{d_2}} \sup_{P^{(n)}} [\ell(\widehat{\dim}_n, \dim(P))] \\ & \geq \left(C_{d_1, d_2, K_I}^{(13)}\right)^n \tau_\ell^{(d_2 - d_1)n} \min\left\{\tau_\ell^{-2(d_2 - d_1) + 1} n^{-2}, 1\right\}^{(d_2 - d_1)n} \end{aligned}$$

Hence by plugging in $d_1 = 1$ and $d_2 = 2$, the minimax rate R_n in (2.6) is lower bounded as in (D.1), as

$$\inf_{\widehat{d}} \sup_{P \in \mathcal{P}} \mathbb{E}_{P^{(n)}}[\ell(\widehat{d}_n, d(P))] \geq \left(C_{K_I}^{(15)}\right)^n \tau_\ell^n \min\{\tau_\ell^{-3} n^{-2}, 1\}^n$$

with $C_{K_I}^{(15)} = C_{d_1=1, d_2=2, K_I}^{(13)}$.

□